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Transformation Methods for Solving Nonlinear Field Equations

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The collection of extended canonical transformations of first-order contact manifolds is studied. This collection is shown to form a group under target-source composition and to contain the group of all first prolongations of point transformation of the underlying graph space and all isogroups of completely integrable horizontal ideals. Extended canonical transformations are compared and contrasted with Bäcklund transformations. These results are used to construct an extended Hamilton-Jacobi method for systems of nonlinear PDE. The collection of all extended canonical transformations is also shown to contain infinitely many one-parameter families of transformations, but there is no Lie group structure that contains these one-parameter families, in general. Conditions are obtained under which a one-parameter family of extended canonical transformations will map a solution of the fundamental ideal that characterizes a given system of PDE into a one-parameter family of solutions. These results are applied to the Ω-Gordon equation $\partial_X \partial_t \phi = \Omega(\phi)$ and to the Navier-Stokes equations.

1. STATEMENT OF THE PROBLEM

We will deal exclusively in this paper with systems of partial differential equations (PDE) with n independent variables and N dependent variables of the form

$$h_a\left(x^i, \phi^\alpha(x^j), \frac{\partial \phi^\alpha}{\partial x^i}\right) = \frac{d}{dx^k} W_a^k\left(x^i, \phi^\alpha(x^j), \frac{\partial \phi^\alpha}{\partial x^i}\right), \qquad 1 \le a \le r \quad (1.1)$$

namely, equations of balance. The underlying geometric structure for such systems is an $(n+N+nN)$-dimensional *contact manifold* K with local coordinates $\{x^i, q^\alpha, y_i^\alpha | 1 \le i \le n, 1 \le \alpha \le N\}$ and *contact 1-forms*

$$C^\alpha = dq^\alpha - y_i^\alpha \, dx^i, \qquad 1 \le \alpha \le N \quad (1.2)$$

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that give rise to the closed *contact ideal*

$$\mathscr{C} = I\{C^\alpha, dC^\alpha | 1 \le \alpha \le N\} \tag{1.3}$$

The balance equations (1.1) can then be encoded by the *balance n-forms*

$$B_a = h_a(x^j, q^\alpha, y_j^\alpha)\mu - dW_a^i(x^j, q^\alpha, y_i^\alpha) \wedge \mu_i \tag{1.4}$$

where $\mu = dx^1 \wedge \cdots \wedge dx^n$ is the volume element on the base manifold $\mathbb{R}^n$, and where $\mu_i = \partial_i \lrcorner \mu$ are the conjugate basis elements of $\Lambda^{n-1}(\mathbb{R}^n)$ that satisfy $d\mu_i = 0$, $dx^j \wedge \mu_i = \delta_i^j \mu$. These structures combine to give the closed *fundamental ideal*

$$\mathscr{I} = I\{C^\alpha, dC^\alpha, B_a | 1 \le \alpha \le N, 1 \le a \le r\} \tag{1.5}$$

because $dB_a \equiv 0 \bmod \mathscr{C}$ and $\mathscr{C}$ is a closed subideal of $\mathscr{I}$. The set of all *solution maps* of the given system of balance equations has the geometric characterization

$$S(B_a) = \{\Phi \colon D_n \subset \mathbb{R}^n \to K | \Phi^*\mu \ne 0, \Phi^*\mathscr{I} = 0\} \tag{1.6}$$

The fundamental problem associated with a given system of equations of balance is to provide computational algorithms for obtaining elements of $S(B_a)$.

It has been shown in a previous paper (Edelen, 1990) that there are alternative geometric formulations that are effective in providing computational algorithms for solution maps. A summary of the pertinent results is given in the remainder of this section. The reader is referred to Edelen (1990) for the details.

The constructions start by introducing the 1-forms

$$H_i^\alpha = dy_i^\alpha - A_{ij}^\alpha \, dx^j, \qquad 1 \le i \le n, \quad 1 \le \alpha \le N \tag{1.7}$$

where the A's are elements of $\Lambda^0(K)$ that are symmetric in their lower indices. These give rise to a *horizontal ideal*

$$\mathscr{H}[A_{ij}^\alpha] = I\{C^\alpha, H_i^\alpha | 1 \le i \le n, 1 \le \alpha \le N\} \tag{1.8}$$

for each choice of the A's, as the notation indicates. The Cauchy characteristic subspace $\mathscr{H}^*[A_{ij}^\alpha]$ of the horizontal ideal $\mathscr{H}[\mathscr{A}_{ij}^\alpha]$ admits the *canonical system*

$$V_i = \partial_i + y_i^\alpha \partial_\alpha + A_{ij}^\alpha \partial_\alpha^j, \qquad 1 \le i \le n \tag{1.9}$$

as a basis. Here,

$$\partial_i = \frac{\partial}{\partial x^i}, \qquad \partial_\alpha = \frac{\partial}{\partial q^\alpha}, \qquad \partial_\alpha^j = \frac{\partial}{\partial y_j^\alpha}$$

is the natural basis for the Lie algebra $T(K)$ of all vector fields on K. It therefore follows that

$$V_i \lrcorner C^\alpha = 0, \qquad V_i \lrcorner H_j^\alpha = 0 \tag{1.10}$$

A fundamental and very useful property of the vector fields $\{V_i\}$ is that

$$df \equiv V_i\langle f\rangle \, dx^i \bmod \mathscr{H}[A_{ij}^\alpha] \tag{1.11}$$

for any $f \in \Lambda^0(K)$.

An arbitrary horizontal ideal will not, in general, have annihilating maps whose graphs are n-dimensional, while the graph of a solution map Φ of the fundamental ideal must be n-dimensional and satisfy $\Phi^*\mu \neq 0$. This difficulty is overcome by the requirement that $\mathscr{H}[A_{ij}^\alpha]$ be *completely integrable.* This is the case if and only if

$$[\![V_i, V_j]\!] = 0 \tag{1.12}$$

which is a complicated system of nonlinear, first-order partial differential equations that the functions A_{ij}^α must satisfy. The collection of all completely integrable horizontal ideals is denoted by $\mathfrak{H}(K)$. When (1.12) are satisfied, the system of linear first-order PDE

$$V_i\langle g\rangle = 0, \qquad 1 \le i \le n \tag{1.13}$$

admits a complete system of $N + nN$ independent primitive integrals $\{g_\Sigma(x^i, q^\alpha, y_i^\alpha) | 1 \le \Sigma \le N + nN\}$, and the space K is foliated locally (i.e., in a neighborhood of any point of K) by n-dimensional leaves with the implicit representations

$$g_\Sigma(x^j, q^\alpha, y_i^\alpha) = k_\Sigma, \qquad 1 \le \Sigma \le N + nN, \qquad \{k_\Sigma\} \in \mathbb{R}^{N+nN} \tag{1.14}$$

A parametric representation of the leaf of this foliation that passes through a given point $P_0 \in K$ can be obtained by sequential integration of the orbital equations for the vector fields $\{V_i | 1 \le i \le n\}$ starting from the point P_0. This parametric representation gives rise to a map Ψ: $D_n \subset \mathbb{R}^n \to K$ such that

$$\Psi^*\mu \neq 0, \qquad \Psi^*C^\alpha = 0, \qquad \Psi^*H_i^\alpha = 0 \tag{1.15}$$

Such a map Ψ thus gives $\Psi^*\mathscr{H}[A_{ij}^\alpha] = 0$ and $\Psi^*\mathscr{C} = 0$, and hence Ψ will be a solution map of the fundamental ideal if $\Psi^*B_a = 0$. Explicit conditions are given in Edelen (1990) in order that the conditions $\Psi^*B_a = 0$ be satisfied. When these conditions are met, solution maps of the given system of balance equations can be obtained by sequential integration of systems of autonomous ordinary differential equations. These solution maps will necessarily satisfy the constraints $\Psi^*H_i^\alpha = 0$, and hence they will only constitute a subset of the collection of all solution maps of the fundamental ideal. It can be shown, however, that the graph of any solution of the fundamental ideal is the leaf of the foliation of K that is generated by some completely integrable horizontal ideal. The solution set of a fundamental ideal can thus be exhausted by considering all completely integrable horizontal ideals.

2. EXTENDED CANONICAL TRANSFORMATIONS

The difficulty inherent in this method is that of obtaining functions $\{A^{\alpha}_{ij}(x^k, q^{\alpha}, y^{\alpha}_i)\}$ that are symmetric in the lower two indices and satisfy the conditions $[\![V_i, V_j]\!] = 0$. One way around this problem is to consider transformations $S: K \to {}'K$ that map the contact manifold K with horizontal ideal $\mathscr{H}[A^{\alpha}_{ij}]$ onto a replica ${}'K$ with horizontal ideal ${}'\mathscr{H}[{}'A^{\alpha}_{ij}]$. Since mappings and exterior differentiation commute, ${}'\mathscr{H}[{}'A^{\alpha}_{ij}]$ will be completely integrable if $\mathscr{H}[A^{\alpha}_{ij}]$ is completely integrable. Transformations with these properties provide us with the means of computing many collections of ${}'A$'s that satisfy $[\![{}'V_i, {}'V_j]\!] = 0$ from any one collection of A's that satisfy $[\![V_i, V_j]\!] = 0$. The following definition is slightly changed from that given in Section 16 of Edelen (1990). The changes have been introduced to provide the basis for certain distinctions that will prove to be important in later sections of this paper. Let $\mathrm{Diff}(K, {}'K)$ denote the pseudogroup of diffeomorphisms of an open set of K onto an open set in ${}'K$.

Definition 2.1. A map $S: K \to {}'K$ that belongs to $\mathrm{Diff}(K, {}'K)$ is an *extended canonical transformation* if and only if there exists a completely integrable horizontal ideal $\mathscr{H}[A^{\alpha}_{ij}]$ of $\Lambda(K)$ such that

$$S^* \, {}'\mathscr{H}[{}'A^{\alpha}_{ij}] \subset \mathscr{H}[A^{\alpha}_{ij}] \tag{2.1}$$

in which case $\mathscr{H}[A^{\alpha}_{ij}]$ is the *source* of S and ${}'\mathscr{H}[{}'A^{\alpha}_{ij}]$ is the *target* of S. The collection of all extended canonical transformations is denoted by

$$\mathrm{ECT} = \{S \in \mathrm{Diff}(K, {}'K) \mid S^* \, {}'\mathscr{H}[{}'A^{\alpha}_{ij}] \subset \mathscr{H}[A^{\alpha}_{ij}], \mathscr{H}[A^{\alpha}_{ij}] \in \mathfrak{H}(K)\} \tag{2.2}$$

The following results have been established in Edelen (1990). Let $\mathscr{H}[A^{\alpha}_{ij}]$ be a completely integrable horizontal ideal of $\Lambda(K)$ and let $\{V_i \mid 1 \le i \le n\}$ be the canonical basis for $\mathscr{H}^*[A^{\alpha}_{ij}]$. A transformation $S \in \mathrm{Diff}(K, {}'K)$, with the presentation

$${}'x^i = s^i(x^j, q^{\beta}, y^{\beta}_j), \qquad {}'q^{\alpha} = s^{\alpha}(x^j, q^{\beta}, y^{\beta}_j), \qquad {}'y^{\alpha}_i = s^{\alpha}_i(x^j, q^{\beta}, y^{\beta}_j) \tag{2.3}$$

is an extended canonical transformation with source $\mathscr{H}[A^{\alpha}_{ij}]$ if and only if

$$\det(V_i\langle s^j \rangle) \neq 0 \tag{2.4}$$

$$V_j\langle s^{\alpha} \rangle = s^{\alpha}_k V_j\langle s^k \rangle \tag{2.5}$$

in which case $'A^\alpha_{ij}$ are determined by

$$^*A^\alpha_{ij} = S^*\,'A^\alpha_{ij} \tag{2.6}$$

$$^*A^\alpha_{km}\,V_j\langle s^m\rangle\,V_i\langle s^k\rangle = V_jV_i\langle s^\alpha\rangle - s^\alpha_k\,V_jV_i\langle s^k\rangle \tag{2.7}$$

and $'\mathscr{H}['A^\alpha_{ij}]$ is completely integrable.

Satisfaction of the conditions (2.4) implies that there exist functions S^i_j such that

$$S^i_j\,V_i\langle s^k\rangle = S^k_i\,V_j\langle s^i\rangle = \delta^k_j \tag{2.8}$$

Accordingly, (2.5) gives the explicit evaluations

$$s^\alpha_j = S^m_j\,V_m\langle s^\alpha\rangle \tag{2.9}$$

Since $S \in \mathrm{Diff}(K, 'K)$, the inverse mapping S^{-1} exists. The relations (2.6) and (2.7) can therefore be used to obtain the explicit evaluations

$$'A^\alpha_{ij} = S^{-1*}(S^m_j S^r_i\{V_m V_r\langle s^\alpha\rangle - S^k_t\,V_k\langle s^\alpha\rangle\,V_m V_r\langle s^t\rangle\}) \tag{2.10}$$

These explicit evaluations show that any extended canonical transformation is determined by specifying the $n+N$ functions $\{s^i, s^\alpha \mid 1 \le i \le n,\ 1 \le \alpha \le N\}$ of the arguments $\{x^j, q^\beta, y^\beta_j\}$ that satisfy the conditions (2.4) and are such that $S \in \mathrm{Diff}(K, 'K)$. This latter requirement is tantamount to the condition

$$\frac{\partial(s^i, s^\alpha, S^m_j\,V_m\langle s^\alpha\rangle)}{\partial(x^k, q^\beta, y^\beta_k)} \ne 0 \tag{2.11}$$

on the Jacobian of the transformation. Functions $\{s^i, s^\alpha\}$ with these properties are referred to as *generating functions* of an extended canonical transformation.

If S is an extended canonical transformation with source $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K)$ and generating functions $\{s^i, s^\alpha\}$, then the canonical basis vectors for $\mathscr{H}^*[A^\alpha_{ij}]$ and $'\mathscr{H}^*['A^\alpha_{ij}]$ are related by

$$S_*\,V_i = \{(V_i\langle s^k\rangle) \circ S^{-1}\}\,'V_k \tag{2.12}$$

It therefore follows that

$$S_*\mathscr{H}^*[A^\alpha_{ij}] = {}'\mathscr{H}^*['A^\alpha_{ij}] \tag{2.13}$$

and hence that the systems $\{g_\Sigma\}$ and $\{'g_\Sigma\}$ of complete, independent, first integrals of the systems $V_i\langle g\rangle = 0$ and $'V_i\langle 'g\rangle = 0$ are related by

$$'g_\Sigma = g_\Sigma \circ S^{-1} \tag{2.14}$$

Thus, an extended canonical transformation with source $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K)$ will carry leaves of the foliation of K generated by $\mathscr{H}[A^\alpha_{ij}]$ into leaves of the foliation of $'K$ generated by the target $'\mathscr{H}['A^\alpha_{ij}]$.

The presentation relations for an extended canonical transformation S, in terms of the generating functions $\{s^i, s^\alpha\}$, are

$$'x^i = s^i(x^i, q^\beta, y_j^\beta), \qquad 'q^\alpha = s^\alpha(x^j, q^\beta, y_j^\beta)$$

If

$$\partial_\beta^j\langle s^i\rangle = 0, \qquad \partial_\beta^j\langle s^\alpha\rangle = 0 \tag{2.15}$$

(i.e., the generating functions do not depend on the y's), then

$$V_i(\{s^j, s^\alpha\}) = Z_i(\{s^j, s^\alpha\}), \qquad Z_i = \partial_i + y_i^\alpha\partial_\alpha \tag{2.16}$$

and (2.5) becomes

$$Z_j\langle s^\alpha\rangle = s_k^\alpha Z_j\langle s^k\rangle \tag{2.17}$$

These relations show that the resulting extended canonical transformation is a *first prolongation* (Pommaret, 1978; Olver, 1986) or *first group extension* (Ovsiannikov, 1982; Ibragimov, 1985) of a point transformation on the $(n+N)$-dimensional graph space G with local coordinates $\{x^i, q^\alpha | 1 \le i \le n, 1 \le \alpha \le N\}$. Now, (2.17) shows that the functions $\{s_i^\alpha\}$ are independent of the choice of the functions $\{A_{ij}^\alpha\}$ and hence such extended canonical transformations are universal with respect to the choice of source $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K)$. This establishes the following result.

Theorem 2.1. Every first prolongation (first group extension) of a point transformation on graph space such that $\det(Z_i\langle s^j\rangle) \neq 0$ is an extended canonical transformation that is universal with respect to the choice of the source horizontal ideal $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K)$. Conversely, an extended canonical transformation is a first prolongation of a point transformation on graph space if and only if the generating functions satisfy the conditions $\partial_\beta^i\langle s^i\rangle = 0$, $\partial_\beta^j\langle s^\alpha\rangle = 0$.

If any one of the quantities $\{\mathscr{Q}\} = \{\partial_\beta^j\langle s^i\rangle, \partial_\beta^j\langle s^\alpha\rangle\}$ is nonzero, then the extended canonical transformation S with generating functions $\{s^i, s^\alpha\}$ is *not* a first prolongation of a point transformation of graph space. An examination of the presentation (2.5) for S shows that at least one of the new independent variables $\{'x^j\}$ or one of the new dependent variables $\{'q^\alpha\}$ will depend on the y's; that is, the image in $'K$ of a point P in graph space will depend on the point P and on the derivative information contained in the values of the y's. This situation is similar in some respects to what occurs with Bäcklund transformations (Bäcklund, 1876; Rogers and Shadwick, 1982). There are significant differences, however. A Bäcklund transformation takes any solution map of the source fundamental ideal into a solution map of the target fundamental ideal, while an extended canonical transformation will only take a solution map of the source fundamental ideal that satisfies the constraints $\Psi^* H_i^\alpha = 0$ into a solution map of the target fundamental ideal that satisfies the constraints $'\Psi^{*\prime} H_i^\alpha = 0$. Further, if any one of the quantities $\{\mathcal{Q}\}$ is not zero, then

$$V_i\langle\{s^i, s^\alpha\}\rangle \neq Z_i\langle\{s^i, s^\alpha\}\rangle$$

for at least one choice of the indices. Equations (2.5) that determine the presentation functions $\{s_i^\alpha\}$ then show that this determination will depend on the evaluations of the functions $\{A_{ij}^\alpha\}$; that is, the presentation functions $\{s_i^\alpha\}$ will depend on the choice of the source $\mathscr{H}[A_{ij}^\alpha]$. In this event, a given choice of the generating functions $\{s^i, s^\alpha\}$ will generate a different extended canonical transformation for each choice of the source $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K)$. This explicit dependence on the source horizontal ideal, for extended canonical transformations that are not first prolongations, is what distinguishes extended canonical transformations from most collections of transformations studied in the literature. In particular, it is evident that prolongation methods in the attendant jet bundle formulations cannot be used because they would restrict us to the subset of extended canonical transformations that are first prolongations.

3. EXAMPLES WITH $n = 2$

In the interests of simplicity, we restrict the considerations to cases where $n = 2$ and $N = 1$. This allows us to use the simplified system of local coordinates $\{x, t, q, y_x, y_t\}$ for K. We start with

$$'x = x, \qquad 't = t, \qquad 'q = q + a\, e^t (y_x)^2 \tag{3.1}$$

and a completely integrable source horizontal ideal whose associated canonical system is

$$V_i = \partial_i + y_i \partial_q + k_{ij}\partial^j, \qquad k_{ij} = k_{ji}, \qquad dk_{ij} = 0 \tag{3.2}$$

that is, $A_{ij}^1 = k_{ij}$, where the k's are constants. The relations (2.9) then give

$$'y_x = y_x + 2a y_x k_{xx}\, e^t, \qquad 'y_t = a\, e^t (y_x)^2 + y_t + 2a k_{xt} y_x\, e^t \tag{3.3}$$

and hence (3.1) and (3.3) show that this transformation belongs to $\text{Diff}(K, 'K)$. The presence of the k's in the relations (3.3) explicitly shows the dependence of the resulting extended canonical transformation on the choice of the source A's because $A_{ij}^1 = k_{ij}$. The $'A$'s are then determined by (2.10), and we obtain

$$\begin{aligned} 'A_{xx}^1 &= (1 + 2a k_{xx}\, e^t) k_{xx} \\ 'A_{xt}^1 &= 2a k_{xx}\, e^t \frac{'y_x}{1 + 2a k_{xx}\, e^t} + (1 + 2a k_{xx}\, e^t) k_{xt} \\ 'A_{tt} &= \frac{a\, e^t ('y_x)^2}{(1 + 2a k_{xx}\, e^t)^2} + \frac{4a k_{xt}\, e^t\, 'y_x}{1 + 2a k_{xx}\, e^t} + 2a (k_{xt})^2\, e^t \end{aligned} \tag{3.4}$$

A direct calculation easily shows that $'\mathscr{H}['A^1_{ij}]$ is completely integrable.

The vector field

$$U = -2t\partial_x + xq\partial_q + (q + xy_x)\partial^x + (2y_x + xy_t)\partial^t \tag{3.5}$$

on K generates a local one-parameter Lie group of prolongations of a one-parameter local Lie group of point transformations of the graph space G with local coordinates $\{x, t, q\}$. It therefore generates a one-parameter group of extended canonical transformations

$$\begin{aligned} S_U(\tau)|'x = x - 2t\tau, \qquad 't = t, \qquad 'q = q \exp(x\tau - t\tau^2) \\ 'y_x = (y_x + q\tau) \exp(x\tau - t\tau^2) \\ 'y_t = (y_t + 2y_x\tau + q\tau^2) \exp(x\tau - t\tau^2) \end{aligned} \tag{3.6}$$

with parameter $\tau \in \mathbb{R}$ for every choice of the source horizontal ideal (see Theorem 2.1). The simplest source with $A^1_{ij} = 0$ is therefore used to compute a one-parameter family of completely integrable targets $'\mathscr{H}['A^1_{ij}]$ by using (2.10). This gives the evaluations

$$\begin{aligned} 'A^1_{xx} = \tau 'y_x, \qquad 'A^1_{xt} = \tau('y_t - \tau\, 'y_x + \tau^2\, 'q) \\ 'A^1_{tt} = \tau^2(2\, 'y_t - 4\tau 'y_x + 3\tau^2\, 'q) \end{aligned} \tag{3.7}$$

for each value of the parameter $\tau \in \mathbb{R}$. A direct calculation verifies that the canonical basis $\{'V_x, 'V_t\}$ for $'\mathscr{H}^*['A^1_{ij}]$ satisfies $[\!['V_x, 'V_t]\!] = 0$. Although the collection $S_U(\tau)$ forms a one-parameter group of extended canonical transformations, these results show that the induced action of $S_U(\tau)$ on $\mathfrak{H}(K)$ is not that of a one-parameter Lie group; simply note that

$$\frac{d}{d\tau}('A^1_{xx}) = 'y_x + \tau \frac{d}{d\tau}('y_x) = 'y_x + \tau('q + 'x'y_x)$$

This fact will have important ramifications throughout the remainder of these discussions.

These results give us procedures for constructing systems of partial differential equations for which we can construct large families of solutions. For the purposes of this discussion, we will keep $n = 2$ and the independent variables $\{x, t\}$, but relax the requirement that $N = 1$. Let $\{'h^\alpha | 1 \le \alpha \le N\}$ be a system of N elements of $\Lambda^0('K)$. In order to make matters specific, we consider the balance 2-forms

$$'B^\alpha = 'h^\alpha\, d'x \wedge d't - d'y^\alpha_t \wedge d't, \qquad 1 \le \alpha \le N \tag{3.8}$$

that characterize the system of PDE (coupled inhomogeneous wave equations in characteristic coordinates)

$$\frac{\partial^2 \phi^\alpha}{\partial' x\, \partial' t} = h^\alpha\left('x, 't, \phi^\beta, \frac{\partial \phi^\beta}{\partial' x}, \frac{\partial \phi^\beta}{\partial' t}\right) \tag{3.9}$$

Previously established results show that

$$'B^\alpha \equiv 'F^\alpha\, d'x \wedge d't \bmod '\mathscr{H}['A^\alpha_{ij}] \tag{3.10}$$

with

$$'F^\alpha = 'h^\alpha - 'V_x\langle 'y^\alpha_t\rangle = 'h^\alpha - 'A^\alpha_{xt} \tag{3.11}$$

Now (see Edelen, 1990), $'\mathscr{H}['A^\alpha_{ij}]$ is special [i.e., belongs to $\mathfrak{H}_s('K)$] if the $'A$'s are generated in the manner specified above for some $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K)$ and the functions $'h^\alpha('x^j, 'q^\beta, 'y^\beta_j)$ have the determinations

$$'h^\alpha = 'A^\alpha_{xt} \tag{3.12}$$

In this event, every leaf of the foliation generated by $'\mathscr{H}['A^\alpha_{ij}]$ is the graph of a solution map of the given system of PDE. As an example of this construction, let us set $N = 1$ and use the $'A$'s that are given by (3.7). Use of (3.12) shows that we can obtain solutions of the PDE (3.9) when $'h^1$ is given by

$$'h^1 = \tau('y_t - \tau\, 'y_x + \tau^2\, 'q) \tag{3.13}$$

that is,

$$\frac{\partial^2 \phi}{\partial' x\, \partial' t} = \tau\left(\frac{\partial \phi}{\partial' t} - \tau \frac{\partial \phi}{\partial' x} + \tau^2 \phi\right) \tag{3.14}$$

for any given numerical value of the parameter τ. Since this process started with the source horizontal ideal such that $A^1_{ij} = 0$, a complete set of primitive integrals of the orbital equations of $\{V_x, V_t\}$ is

$$g = q - xy_x - ty_t, \qquad g_x = y_x, \qquad g_t = y_t \tag{3.15}$$

Since any extended canonical transformation maps primitive integrals onto primitive integrals, and the inverse of the transformation (3.6) is given by

$$\begin{gathered} x = 'x + 2\, 't\tau, \qquad t = 't, \qquad q = 'q \exp(-'x\tau - 't\tau^2) \\ y_x = ('y_x - 'q\tau) \exp(-'x\tau - 't\tau^2) \\ y_t = ('y_t - 2'y_x\tau + 'q\tau^2) \exp(-'x\tau - 't\tau^2) \end{gathered} \tag{3.16}$$

a complete system of primitive integrals of the canonical system $\{'V_x, 'V_t\}$ is given by

$$
\begin{aligned}
'g &= 'q \exp(-'x\tau - 't\tau^2) - ('x + 2't\tau)'g_x - 't'g_t \\
'g_x &= ('y_x - 'q\tau) \exp(-'x\tau - 't\tau^2) \\
'g_t &= ('y_t - 2'y_x\tau + 'q\tau^2) \exp(-'x\tau - 't\tau^2)
\end{aligned} \tag{3.17}
$$

Solutions of (3.14) are therefore given by

$$
'g = k_1, \qquad 'g_x = k_2, \qquad 'g_t = k_3 \tag{3.18}
$$

for every choice of the constants $\{k_1, k_2, k_3\}$ because $'\mathcal{H}['A^1_{ij}] \in \mathfrak{H}_s('K_1)$ i.e., we have

$$
'F_1 = 'h^1 - 'A^1_{xt} = 0
$$

If we use the extended canonical transformation given by (3.1)-(3.4) with $k_{xt} = 0$, then the balance 2-form

$$
'B_1 = 'h_1\, '\mu - d'y_t \wedge '\mu_t \tag{3.19}
$$

encodes the PDE

$$
\frac{\partial^2 \phi}{\partial' t\, \partial' t} = 'h_1 \tag{3.20}
$$

and the corresponding $'F_1$ is given by

$$
'F_1 = 'h_1 - 'A^1_{tt} \tag{3.21}
$$

Thus, if $'h_1$ has the evaluation

$$
'h_1 = \frac{a\, e^t}{(1 + 2ak_{xx}\, e^t)^2} ('y_x)^2 \tag{3.22}
$$

then (3.4) and (3.21) show that $'F_1 = 0$, and hence $'\mathcal{H}['A^1_{ij}]$ belongs to $\mathfrak{H}_s('K)$. Every leaf of the foliation generated by $'\mathcal{H}['A^1_{ij}]$ will thus be a solution of the PDE

$$
\frac{\partial^2 \phi}{\partial' t\, \partial' t} = \frac{a\, e^t}{(1 + 2ak_{xx}\, e^t)^2} \left(\frac{\partial \phi}{\partial' x}\right)^2 \tag{3.23}
$$

The easiest way of obtaining the leaves of the foliation is to invert the extended canonical transformation and use the results to express the primitive integrals $\{g, g_x, g_t\}$ of $\{V_x, V_t\}$ to obtain the primitive integrals of $\{'V_x, 'V_t\}$. If we denote this inverse transformation by S^{-1}, then

$$
'g = g \circ S^{-1}, \qquad 'g_x = g_x \circ S^{-1}, \qquad 'g_t = g_t \circ S^{-1} \tag{3.24}
$$

The solutions are then given in implicit form by

$$'g = k_1, \qquad 'g_x = k_2, \qquad 'g_t = k_3 \tag{3.25}$$

for various choices of the constants $\{k_1, k_2, k_3\}$.

4. GROUP PROPERTIES

A first-order contact manifold K admits the Cartesian product decomposition $K = G \times \mathbb{R}^{nN}$, where G is the associated graph space with local coordinates $\{x^i, q^\alpha | 1 \leq i \leq n,\ 1 \leq \alpha \leq N\}$. Let Diff$(G, 'G)$ denote the group of local diffeomorphisms from the graph space G to the graph space $'G$ with local coordinates $\{'x^i, 'q^\alpha\}$. The Lie algebra of Diff$(G, 'G)$ is $T(G)$, and $T(G)$ lifts to the Lie subalgebra $\mathbf{pr}^{(1)}(T(G))$ of $T(K)$ of first prolongations. Accordingly, Diff$(G, 'G)$ lifts to the Lie pseudogroup $\mathbf{PR}^{(1)}(G, 'G) = \exp\{\mathbf{pr}^{(1)}(T(G))\}$ of all locally regular prolongations of regular point transformations on graph space [see Pommaret (1978) for a discussion of why only a pseudogroup structure is obtained]. The presentation relations for elements of $\mathbf{PR}^{(1)}(G, 'G)$ have the form (Olver, 1986; Pommaret, 1978)

$$'x^i = P^i(x^j, q^\beta), \qquad 'q^\alpha = P^\alpha(x^j, q^\beta), \qquad 'y_i^\alpha = P_i^\alpha(x^j, q^\beta, y_j^\beta) \tag{4.1}$$

where the functions $\{P_i^\alpha\}$ are determined by

$$Z_i\langle P^\alpha \rangle = P_k^\alpha Z_i\langle P^k \rangle, \qquad Z_i = \partial_i + y_i^\alpha \partial_\alpha \tag{4.2}$$

The regularity condition is contained in the requirement

$$\det(Z_i\langle P^j \rangle) \neq 0 \tag{4.3}$$

Noting that $V_i\langle\{P^i, P^\alpha\}\rangle = Z_i\langle\{P^i, P^\alpha\}\rangle$, where $\{V_i | 1 \leq i \leq n\}$ is the canonical basis for any $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K)$, Theorem 2.1 shows that any element of the Lie pseudogroup $\mathbf{PR}^{(1)}(G, 'G)$ belongs to ECT for every source $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K)$. We have therefore established the following result.

Theorem 4.1. The Lie pseudogroup $\mathbf{PR}^{(1)}(G, 'G)$ of all locally regular first prolongations of point transformations on graph space is properly contained in ECT, and $\mathbf{PR}^{(1)}(G, 'G)$ is universal with respect to the choice of the source $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K)$.

This theorem is the group-theoretic analog of Theorem 2.1. Since the pseudogroup of prolongations is properly contained in ECT, it follows that the restriction to prolongations would eliminate a large part of ECT.

Let $\mathscr{I}$ be a fundamental ideal of $\Lambda(K)$ that characterizes a given system of PDE, let iso$[\mathscr{I}]$ be the Lie algebra of isovectors of the fundamental ideal $\mathscr{I}$, and let ISO$[\mathscr{I}] = \exp\{\text{iso}[\mathscr{I}] \cap \mathbf{pr}^{(1)}(T(G))\}$ be the Lie pseudogroup of symmetry transformations of the fundamental ideal. Since we have

$\text{ISO}[\mathscr{I}] \subset \mathbf{PR}^{(1)}(G, 'G)$, Theorem 4.1 shows that $\text{ISO}[\mathscr{I}]$ is a Lie pseudogroup that is contained in ECT and that $\text{ISO}[\mathscr{I}]$ is universal with respect to the choice of the source horizontal ideal. Any solution map Ψ of the fundamental ideal can be embedded in the Lie pseudogroup $\text{ISO}[\mathscr{I}] \circ \Psi$ of solution maps of the fundamental ideal. On the other hand, it was shown in Edelen (1990) that any smooth solution map Ψ of the fundamental ideal is the leaf map of a leaf of the foliation generated by some completely integrable horizontal ideal $\mathscr{H}[A_{ij}^\alpha]$. Let J be any element of $\text{ISO}[\mathscr{I}]$; then $J \circ \Psi$ is a solution map of the fundamental ideal. Theorem 4.1 then shows that $J \circ \Psi$ is a leaf map of the completely integrable horizontal ideal $'\mathscr{H}['A_{ij}^\alpha]$ that is the target of the extended canonical transformation J with source $\mathscr{H}[A_{ij}^\alpha]$. If we were to start with a different solution map Φ of the fundamental ideal that is a leaf map of the completely integrable horizontal ideal $\mathscr{H}[\tilde{A}_{ij}^\alpha]$, then $J \circ \Phi$ is a solution map of the fundamental ideal that is also a leaf map of the foliation generated by $'\mathscr{H}['A_{ij}^\alpha]$. Here, $'\mathscr{H}['\tilde{A}_{ij}^\alpha]$ is the target of the extended canonical transformation J with source $\mathscr{H}[\tilde{A}_{ij}^\alpha]$. The reason why these constructions work is the fact that any prolongation of a point transformation of graph space is an extended canonical transformation that is universal with respect to the choice of the source horizontal ideal.

The collection $\text{ISO}[A_{ij}^\alpha] \subset T(K)$ is the Lie algebra of isovectors of the completely integrable horizontal ideal $\mathscr{H}[A_{ij}^\alpha]$ (see Edelen, 1990). It thus follows that every transformation in the Lie pseudogroup $\mathbf{ISO}[A_{ij}^\alpha] = \exp\{\text{ISO}[A_{ij}^\alpha]\}$ maps $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K)$ into itself. The definition of ECT shows that $\mathbf{ISO}[A_{ij}^\alpha]$ is a Lie pseudogroup of elements of ECT with source and target $\mathscr{H}[A_{ij}^\alpha]$. This observation establishes the following result.

Theorem 4.2. The Lie pseudogroup $\mathbf{ISO}[A_{ij}^\alpha]$ is contained in ECT for every $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K)$, and all elements of $\mathbf{ISO}[A_{ij}^\alpha]$ have $\mathscr{H}[A_{ij}^\alpha]$ as both source and target.

These results suggest that an investigation of possible group properties of ECT will be useful. Let S_1 be an extended canonical transformation with source $\mathscr{H}[A_{ij}^\alpha]$ and target $'\mathscr{H}['A_{ij}^\alpha]$. We therefore have

$$'x^i = s^i(x^j, q^\beta, y_j^\beta), \qquad 'q^\alpha = s^\alpha(x^j, q^\beta, y_j^\beta)$$
$$'y_i^\alpha = s_i^\alpha(x^j, q^\beta, y_j^\beta) \tag{4.4}$$

with

$$V_j\langle s^\alpha \rangle = s_i^\alpha V_j\langle s^i \rangle \tag{4.5}$$

Let S_2 be an extended canonical transformation with source $'\mathscr{H}['A_{ij}^\alpha]$ and target $''\mathscr{H}[''A_{ij}^{\alpha}]$. This gives us

$$''x^i = 's^i('x^j, 'q^\beta, 'y_j^\beta), \qquad ''q^\alpha = 's^\alpha('x^j, 'q^\beta, 'y_j^\beta),$$
$$''y_i^\alpha = 's_i^\alpha('x^j, 'q^\beta, 'y_j^\beta) \tag{4.6}$$

with

$$'V_j\langle 's^\alpha\rangle = 's_i^\alpha \, 'V_j\langle 's^i\rangle \tag{4.7}$$

We can then compose S_2 with S_1 to obtain the transformation $S_{21} = S_2 \circ S_1$ with source $\mathscr{H}[A_{ij}^{\alpha}]$ and target $''\mathscr{H}[''A_{ij}^{\alpha}]$, because the target of S_1 coincides with the source of S_2. We refer to this form of composition of elements of ECT as *target–source* composition. The group property of ECT will then follow if we can show that S_{21} is an extended canonical transformation. It is obvious from the presentation relations for S_1 and S_2 given above that the generating functions for the transformation S_{21} will be the compositions of the generating functions for S_2 with those for S_1; i.e., that

$$''s^i = 's^i(s^j, s^\beta, s_j^\beta), \qquad ''s^\alpha = 's^\alpha(s^j, s^\beta, s_j^\beta) \tag{4.8}$$

Theorem 2.1 then shows that S_{21} is an extended canonical transformation if and only if

$$V_k\langle 's^\alpha(s^j, s^\beta, s_j^\beta)\rangle = ''s_i^\alpha \, V_k\langle 's^i(s^j, s^\beta, s_j^\beta)\rangle \tag{4.9}$$

because S_{21} has $\mathscr{H}[A_{ij}^{\alpha}]$ as source. However, (2.12) gives the relations

$$(S_1)_* V_i = \{(V_i\langle s^k\rangle) \circ S_1^{-1}\} \, 'V_k \tag{4.10}$$

which serve to show that (4.5), (4.7), and (4.9) are consistent. We have therefore established the following result.

Theorem 4.3. The collection ECT forms a group under target–source composition.

The reader is warned that ECT does not form a Lie pseudogroup, because the composition law is target–source composition rather than unrestricted composition. The group ECT does contain the infinite-dimensional Lie pseudogroup $\mathbf{PR}^1(G, 'G)$ because $\mathbf{PR}^1(G, 'G)$ is a Lie pseudogroup that is universal with respect to choice of source. It also contains the infinite-dimensional Lie pseudogroup $\mathbf{ISO}[A_{ij}^{\alpha}]$ because $\mathbf{ISO}[A_{ij}^{\alpha}]$ is a Lie pseudogroup for which the source and target always coincide.

Let $\mathscr{H}[0]$ denote the completely integrable horizontal ideal that obtains for the choice $A_{ij}^{\alpha} = 0$; that is, when $V_i = Z_i$. This is certianly the simplest completely integrable horizontal ideal that can be written down. The following result shows that any completely integrable horizontal ideal can be constructed from $\mathscr{H}[0]$ by the action of appropriate extended canonical transformations.

Theorem 4.4. There exists an element of ECT with source $\mathscr{H}[A^\alpha_{ij}]$ and target $'\mathscr{H}[0]$ for any $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K)$, provided the base of K is a sufficiently small open subset of $\mathbb{R}^n$, and this transformation leaves the base manifold invariant. Hence, $\mathfrak{H}('K)$ is the orbit of $\mathscr{H}[0]$ under the action of ECT, and this action is multiply transitive.

Proof. Let $\mathscr{H}[A^\alpha_{ij}]$ be a generic element of $\mathfrak{H}(K)$ and let $\{g^\alpha, g^\alpha_i | 1 \le \alpha \le N, 1 \le i \le n\}$ be a complete, independent system of primitive integrals of the system $\{V_i\langle g\rangle = 0 | 1 \le i \le n\}$, where $\{V_i\}$ is the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$. Consider the transformation S with source $\mathscr{H}[A^\alpha_{ij}]$ and generating functions

$$s^i = x^i, \qquad s^\alpha = g^\alpha + (x^i - x^i_0) g^\alpha_i \tag{4.11}$$

Since $V_i\langle\{g^\alpha, g^\alpha_i\}\rangle = 0$ and $V_i\langle s^j\rangle = \delta^j_i$, the presentation of S is

$$'x^i = x^i, \qquad 'q^\alpha = g^\alpha + (x^i - x^i_0) g^\alpha_i, \qquad 'y^\alpha_i = g^\alpha_i \tag{4.12}$$

and hence S leaves the base manifold D_n invariant. The transformation S will thus be an extended canonical transformation provided $S \in \text{Diff}(K, 'K)$, that is, provided

$$\frac{\partial('x^i, 'q^\alpha, 'y^\alpha_i)}{\partial(x^j, q^\beta, y^\beta_j)} = \frac{\partial(g^\alpha + (x^k - x^k_0) g^\alpha_k, g^\alpha_i)}{\partial(q^\beta, y^\beta_j)} \neq 0$$

The independence of the primitive integrals $\{g^\alpha, g^\alpha_i\}$ then shows that this condition will be satisfied for all values of $\{x^k\}$ such that $\|x^k - x^k_0\|$ is sufficiently small. Hence, S is an extended canonical transformation provided the base manifold of K is a sufficiently small open subset of $\mathbb{R}^n$. It then follows from (2.7) that we have $^*A^\alpha_{ij} = 0$, and hence $'A^\alpha_{ij} = 0$. The target of S is therefore $'\mathscr{H}[0]$. Reversing the roles of source and target, and using the group property of ECT, it follows that there exists an extended canonical transformation with source $\mathscr{H}[0]$ and target $'\mathscr{H}['A^\alpha_{ij}]$, for each $'\mathscr{H}['A^\alpha_{ij}] \in \mathfrak{H}('K)$. This shows that $\mathfrak{H}('K)$ is the orbit of $\mathscr{H}[0]$ under the action of ECT. That this orbit is multiply transitive follows from the group property of ECT and the fact that ECT contains the Lie pseudogroups **ISO**$[A^\alpha_{ij} = 0]$ and **ISO**$['A^\alpha_{ij}]$ for every $'\mathscr{H}['A^\alpha_{ij}] \in \mathfrak{H}('K)$. ■

A transformation S of the group ECT takes the source ideal $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K)$ into the target ideal $'\mathscr{H}['A^\alpha_{ij}] \in \mathfrak{H}('K)$. If we then drop the primes on the names of the quantities in $'K$, the space $'K$ becomes indistinguishable from the space K, and the target ideal $'\mathscr{H}['A^\alpha_{ij}]$ becomes a new ideal of $\Lambda(K)$ that belongs to $\mathfrak{H}(K)$. We refer to this action of ECT on $\mathfrak{H}(K)$ as *pulled back action.* Theorem 4.4 can then be used to show that *the collection*

$\mathfrak{H}(K)$ of all completely integrable horizontal ideals is the orbit of $\mathfrak{H}[0]$ under the pulled back action of the group of extended canonical transformations. A complete characterization and computationally feasible construction of $\mathfrak{H}(K)$ has thus been obtained. The reader is warned, however, that the pull back action of the group ECT on $\mathfrak{H}(K)$ is not a Lie group action. This is explicitly shown in the second example given in Section 3, where the elements of ECT constitute a one-parameter Lie subgroup of $\mathbf{PR}^1(G, 'G)$.

5. AN EXTENDED HAMILTON–JACOBI METHOD

The results established in Theorem 4.4 suggest that we concentrate our attention on transformations that leave the base manifold invariant. An extended canonical transformation S with these properties has the presentation

$$S|'x^i = s^i = x^i, \qquad 'q^\alpha = s^\alpha(x^j, q^\beta, y_j^\beta) \tag{5.1}$$

$$'y_i^\alpha = V_i\langle s^\alpha(x^j, q^\beta, y_j^\beta)\rangle \tag{5.2}$$

where

$$V_i = \partial_i + y_i^\alpha \partial_\alpha + A_{ij}^\alpha(x^k, q^\beta, y_k^\beta)\partial_\alpha^j, \qquad 1 \le i \le n \tag{5.3}$$

is the canonical basis for $\mathscr{H}^*[A_{ij}^\alpha]$ that is the Cauchy characteristic module of the source horizontal ideal $\mathscr{H}[A_{ij}^\alpha]$. The induced transformations of the A's are given by

$$^*A_{ij}^\alpha = S^{*\prime}A_{ij}^\alpha = V_iV_j\langle s^\alpha(x^k, q^\beta, y_k^\beta)\rangle \tag{5.4}$$

and the fact that $S \in \mathrm{Diff}(K, 'K)$ gives the local invertibility condition

$$\frac{\partial(s^\alpha, V_i\langle s^\beta\rangle)}{\partial(q^\gamma, y_k^\delta)} \ne 0 \tag{5.5}$$

Suppose that we drop the local invertibility condition (5.5). In this event, the range of S in $'K$ will be at least of dimension n in view of the requirements $'x^i = x^i$. In fact, even if we were to take $s^\alpha = 0$ for all values of the index α, S would map K onto the n-dimensional manifold in $'K$ given by $'q^\alpha = 0$, $'y_i^\alpha = 0$. Hence, if Ψ is a regular map from an open set J_n of $\mathbb{R}^n$ into K, then $S \circ \Psi$ will be a map from J_n into $'K$ with an n-dimensional range such that $(S \circ \Psi)^{*\prime}\mu \ne 0$. The map $S \circ \Psi$ could thus serve as a solution map of the balance ideal of $\Lambda('K)$ that characterizes a given system of PDE on $'K$.

Let

$$'B_a = 'h_a('x^j, 'q^\beta, 'y_j^\beta)'\mu - d'W_a^i('x^j, 'q^\beta, 'y_j^\beta) \wedge '\mu_i, \qquad 1 \le a \le r \tag{5.6}$$

be the balance n-forms that characterize a given system of PDE on $'K$. If $S: K \to 'K$ is a map with presentation (5.1), (5.2) that does not necessarily satisfy the condition (5.5), then

$$S^{*\prime}B_a = ('h_a \circ S)\mu - d('W^i_a \circ S) \wedge \mu_i, \qquad 1 \le a \le r \tag{5.7}$$

is a well-defined system of r n-forms on K. Accordingly, if $\Psi: J_n \to K$ is a regular map, then $(S \circ \Psi)^{*\prime}B_a$ are well-defined n-forms on J_n. Thus, if $(S \circ \Psi)^{*\prime}B_a = 0$, $1 \le a \le r$, then $S \circ \Psi$ will solve the balance ideal

$$'\mathscr{B}_1['A^\alpha_{ij}] = I\{'C^\alpha, 'H^\alpha_i, 'B_a\} \tag{5.8}$$

of $\Lambda('K)$ provided $(S \circ \Psi)^{*\prime}\mathscr{H}['A^\alpha_{ij}] = 0$. However, the construction of S guarantees that

$$S^{*\prime}\mathscr{H}['A^\alpha_{ij}] \subset \mathscr{H}[A^\alpha_{ij}]$$

and hence $(S \circ \Psi)^{*\prime}\mathscr{H}['A^\alpha_{ij}] = 0$ if $\Psi^*\mathscr{H}[A^\alpha_{ij}] = 0$. Thus, if Ψ is any solution map of the source horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$, then $S \circ \Psi$ will solve the balance ideal $'\mathscr{B}['A^\alpha_{ij}]$ provided S can be chosen so that $(S \circ \Psi)^{*\prime}B_a = 0$.

Definition 5.1. A map

$$\mathrm{HJ}|'x^i = x^i, \qquad 'q^\alpha = J^\alpha(x^j, q^\beta, r^\beta_j), \qquad 'y^\alpha_i = V_i\langle J^\alpha\rangle \tag{5.9}$$

of K to $'K$ with source horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ is an *extended Hamilton-Jacobi map* for a system of PDE that is characterized by the balance n-forms

$$'B_a = 'h_a\, '\mu - d'W^i_a \wedge '\mu_i \tag{5.10}$$

on $'K$ if and only if N *generating functions* $J^\alpha \in \Lambda^0(K)$ can be found such that

$$\mathrm{HJ}^{*\prime}B_a \equiv 0 \bmod \mathscr{H}[A^\alpha_{ij}], \qquad 1 \le a \le r \tag{5.11}$$

Theorem 5.1. Let $\mathscr{H}[A^\alpha_{ij}]$ be a completely integrable horizontal ideal of $\Lambda(K)$. If N functions $J^\alpha \in \Lambda^0(K)$ can be found that satisfy the system of partial differential equations

$$\hat{F}_a = 'h_a \circ \mathrm{HJ} - V_i\langle 'W^i_a \circ \mathrm{HJ}\rangle = 0, \qquad 1 \le a \le r \tag{5.12}$$

where $\mathrm{HJ}: K \to 'K$ has the presentation (5.9), then HJ is a Hamilton-Jacobi map for the system of PDE with balance n-forms (5.10) and $\mathrm{HJ} \circ \Psi$ is a solving map of the balance ideal

$$'\mathscr{B}['A^\alpha_{ij}] = I\{'C^\alpha, 'H^\alpha_i, 'B_a\} \tag{5.13}$$

for any leaf map Ψ of the foliation of K that is generated by $\mathscr{H}[A^\alpha_{ij}]$.

Proof. The congruence

$$'B_a = 'h_a\, '\mu - d'W^i_a \wedge '\mu_i \equiv 'F_a\, '\mu \bmod '\mathscr{H}['A^\alpha_{ij}] \tag{5.14}$$

with

$$'F_a = 'h_a - 'V_i\langle 'W_a^i \rangle$$

was established in Section 6 of Edelen (1990). Now, $V_i\langle 'x^j \rangle = \delta_i^j$ for the map HJ with presentation (5.9), and hence $\mathrm{HJ}_* V_i = 'V_i$ by (2.12). Since $\mathrm{HJ}^*\mathrm{HJ}_* V_i = V_i$ and $\mathrm{HJ}^{*\prime}\mathscr{H}['A_{ij}^\alpha] \subset \mathscr{H}[A_{ij}^\alpha]$, we have

$$\mathrm{HJ}^{*\prime}B_a \equiv \hat{F}_a \mu \bmod \mathscr{H}[A_{ij}^\alpha] \tag{5.15}$$

with

$$\hat{F}_a = 'h_a \circ \mathrm{HJ} - V_i\langle 'W_a^i \circ \mathrm{HJ} \rangle \tag{5.16}$$

If the functions J^α can be chosen such that (5.12) are satisfied, then $\hat{F}_a = 0$ and (5.15) show that we will have $\mathrm{HJ}^{*\prime}B_a \equiv 0 \bmod \mathscr{H}[A_{ij}^\alpha]$. Definition 5.1 thus shows that the map HJ is an extended Hamilton–Jacobi map for the system of PDE characterized by the balance n-forms (5.10). If Ψ is a leaf map of the foliation of K that is generated by $\mathscr{H}[A_{ij}^\alpha]$, then $\Psi^*\mathscr{H}[A_{ij}^\alpha] = 0$ and (5.15) shows that $(\mathrm{HJ} \circ \Psi)^{*\prime}B_a = 0$. Since we necessarily have $\mathrm{HJ}^{*\prime}\mathscr{H}['A_{ij}^\alpha] \subset \mathscr{H}[A_{ij}^\alpha]$, (5.13) shows that $(\mathrm{HJ} \circ \Psi)^{*\prime}\mathscr{B}['A_{ij}^\alpha] = 0$ and hence $\mathrm{HJ} \circ \Psi$ is a solving map of the balance ideal for every leaf map Ψ of the horizontal ideal $\mathscr{H}[A_{ij}^\alpha]$. ■

A Hamilton–Jacobi map pulls the balance n-forms for the system of PDE on $'K$ back to a system of n-forms on K that are trivial mod $\mathscr{H}[A_{ij}^\alpha]$ (i.e., 0-valued mod $\mathscr{H}[A_{ij}^\alpha]$), and hence they become trivially solvable by all leaf maps of the foliation of K that is generated by $\mathscr{H}[A_{ij}^\alpha]$. Since exactly the same circumstances occur in the classical Hamilton–Jacobi theory for Hamiltonian systems of ODE (i.e., the Hamiltonian equations pull back to equations that are trivially integrable), the name "Hamilton–Jacobi" maps seems appropriate.

This method can be applied to any system of PDE for any choice of the source horizontal ideal $\mathscr{H}[A_{ij}^\alpha]$. Since the V's depend on the choice of the A's, the system of equations (5.12) that the J's must satisfy will depend on the choice of the A's. Clearly, some choices of the A's are better than others, both from the standpoint of the ease of solving the system (5.12) and from the standpoint of the ease of solving for the leaf maps. One choice in particular suggests itself; namely $A_{ij}^\alpha = 0$, for in this case $V_i = Z_i$ and the leaf maps are given by

$$x^i = x_0^i + u^i, \qquad q^\alpha = q_0^\alpha + y_{0j}^\alpha u^j, \qquad y_i^\alpha = y_{0i}^\alpha \tag{5.17}$$

Solutions of the given system of PDE, which admit the functions $J^\alpha(x^j, q^\beta, y_j^\beta)$ as generating functions of a Hamilton–Jacobi map, are thus given by

$$'q^\alpha = J^\alpha(x_0^j + u^j,\, q_0^\beta + y_{0j}^\beta u^j,\, y_{0j}^\beta) \tag{5.18}$$

As an example, let us look at the Ω-Gordon equation, where we use the coordinate cover $\{x, t, q, y_x, y_t\}$ for K. The balance 2-form for this equation on $'K$ is given by

$$'B_1 = \Omega('q)'\mu - d'y_x \wedge '\mu_t \tag{5.19}$$

A Hamilton–Jacobi map for this problem must have the presentation

$$\begin{aligned} 'x = x, \qquad 't = t, \qquad 'q = J(x, t, q, y_x, y_t) \\ 'y_x = V_x\langle J\rangle, \qquad 'y_t = V_t\langle J\rangle \end{aligned} \tag{5.20}$$

Let us choose $A^1_{xx} = A^1_{xt} = A^1_{tt} = 0$, so that $V_i = Z_i = \partial_i + y_i\partial_q$. It then follows directly form (5.19) and (5.12) that the function J must satisfy the second-order PDE

$$Z_x Z_t\langle J\rangle = \Omega(J) \tag{5.21}$$

in order for $J(x, t, q, y_x, y_t)$ to be a generating function of a Hamilton–Jacobi map for this problem.

When (5.21) is written out in fully expanded form, we have

$$\partial_x\partial_t J + y_x\partial_t\partial_q J + y_t\partial_x\partial_q J + y_x y_t\partial_q\partial_q J = \Omega(J) \tag{5.22}$$

There are two cases in which this equation can be readily solved. In the first case, J is a function of the variables $\{x, t\}$ only, in which case we obtain $\partial_x\partial_t J = \Omega(J)$; that is, the Ω-Gordon equation itself. If $\phi(x, t)$ is any smooth solution of the Ω-Gordon equation, then $J = \phi(x, t)$ will be a generating function for a Hamilton–Jacobi map. This map has the presentation

$$\text{HJ}|'x = x, \qquad 't = t, \qquad 'q = \phi(x, t), \qquad 'y_x = \partial_x\phi, \qquad 'y_t = \partial_t\phi \tag{5.23}$$

and hence HJ maps all of K onto the two-dimensional surface in $'K$ that is defined by (5.23). Composition of HJ with any leaf map of $\mathcal{H}[A^1_{ij} = 0]$ gives the same two-dimensional surface in $'K$ which is a solution of the Ω-Gordon equation. It is of interest to note in this context that we also obtain the evaluations

$$*A^1_{ij} = V_i V_j\langle J\rangle = \frac{\partial^2\phi(x, t)}{\partial x^i\,\partial x^j}$$

and hence we have

$$'A^1_{ij} = \frac{\partial^2\phi('x, 't)}{\partial' x^i\,\partial' x^j}$$

which are known to be an appropriate choice for the $'A$'s to obtain a solution of the problem as a leaf of a foliation of $'K$.

In the second case, we look for solutions of (5.22) that are independent of x and t; we set $J = \zeta(q, y_x, y_t)$. Equation (5.22) then reduces to the ODE

$$y_x y_t \frac{d^2\zeta}{dq^2} = \Omega(\zeta) \tag{5.24}$$

because the dependence on y_x and y_t is parametric. Equation (5.24) shows that ζ, and hence J, will depend explicitly on the arguments $\{r_x, r_t\}$, and hence the Hamilton–Jacobi map obtained in this case is not the prolongation of a map between graph spaces. This equation admits the first integral

$$y_x y_t \left(\frac{d\zeta}{dq}\right)^2 = k(y_x, u_t) + 2\int_0^{\zeta} \Omega(v)\, dv \tag{5.25}$$

and hence $\zeta(q, y_x, y_t)$ can be obtained by an additional quadrature for reasonable choices of the function Ω. This solution will generate a Hamilton–Jacobi map HJ with the presentation

$$'x = x, \qquad 't = t, \qquad 'q = \zeta(q, y_x, y_t), \qquad 'y_x = y_x \frac{\partial\zeta}{\partial q}, \qquad 'y_t = y_t \frac{\partial\zeta}{\partial q} \tag{5.26}$$

It is easily checked that this map has rank greater than or equal to 4 at all points where $d\zeta/dq \neq 0$, and hence HJ will map K onto a subset of $'K$ of dimension at least 4. These equations show, however, that $'y_x/'y_t = y_x/y_t$, and hence $^*A^1_{ij} = V_i V_j \langle 'q \rangle$ and (5.24) yield

$$'A^1_{xx} = \Omega('q)'y_x/'y_t, \qquad 'A^1_{xt} = \Omega('q), \qquad 'A^1_{tt} = \Omega('q)'y_t/'y_x \tag{5.27}$$

These, however, are exactly the forms of the A's that were found to work for the Ω-Gordon equation in the analysis given in Edelen (1990). The theory tells us that we need only compose HJ with a leaf map Ψ generated by the horizontal ideal $\mathscr{H}[A^1_{ij} = 0]$ in order to obtain a solving map of the balance ideal. Since $x = x_0 + u^1$, $t = t_0 + u^2$, $q = q_0 + y_{0x}u^1 + y_{0t}u^2$, $y_x = y_{0x}$, $y_t = y_{0t}$ is the form taken by any such leaf map of $\mathscr{H}[A^1_{ij} = 0]$, we obtain the solutions

$$'x = u^1, \qquad 't = u^2, \qquad 'q = \zeta(q_0 + y_{0x}u^1 + y_{0t}u^2, y_{0x}, y_{0t}) \tag{5.28}$$

The extended Hamilton–Jacobi method replaces a given system of PDE by a new system, namely (5.12), which will often be worse than the given system with which we started. There are cases, as also happens with the classical Hamilton–Jacobi theory, for which the new system (5.12) will be simpler or easier to solve than the original system. In these cases, the extended Hamilton–Jacobi method can often provide exact solutions to otherwise very complicated problems. The exhaustive property of this extended Hamilton–Jacobi method is established by the following result.

Theorem 5.2. Every smooth (C^3) solution of a given system of equations of balance can be obtained by an extended Hamilton–Jacobi map.

Proof. Let $q^\alpha = \phi^\alpha(x^k)$ be a smooth solution of a given system of equations of balance

$$B_a = h_a\mu - dW^i_a \wedge \mu_i \tag{5.29}$$

and consider the transformation

$$S|'x^i = x^i, \qquad 'q^\alpha = \phi^\alpha(x^k), \qquad 'y^\alpha_i = \partial_i\phi^\alpha(x^k) \tag{5.30}$$

An evaluation of $\hat{F}_a$ by use of (5.12) gives

$$\hat{F}_a = h_a(x^j, \phi^\alpha(x^k), \partial_j\phi^\alpha) - \frac{d}{dx^i} W^i_a(x^j, \phi^\alpha(x^k), \partial_j\phi^\alpha) \tag{5.31}$$

and hence $\hat{F}_a = 0$, $1 \le a \le r$, because $q^\alpha = \phi^\alpha(x^k)$ defines a solution of the balance system (5.47) by hypothesis. The transformation S is therefore an extended Hamilton–Jacobi map with $J^\alpha = \phi^\alpha(x^k)$ by Theorem 5.1. It is of interest to note that we also have

$$'A^\alpha_{ij} = \frac{\partial\phi^\alpha('x^k)}{\partial'x^i\,\partial'x^j} \tag{5.32}$$

for the Hamilton–Jacobi map S, and hence this theorem is the natural complement of Theorem 5.4 of Edelen (1990). ■

6. ONE-PARAMETER FAMILIES OF TRANSFORMATIONS

The action of a one-parameter family $S(\tau): K \to 'K$ of transformations has the presentation

$$'x^i = X^i(x^j, q^\beta, y^\beta_j; \tau), \qquad 'q^\alpha = Q^\alpha(x^j, q^\beta, y^\beta_j; \tau) \tag{6.1}$$
$$'y^\alpha_i = Y^\alpha_i(x^j, q^\beta, y^\beta_j; \tau)$$

We require these transformations to reduce to the identity transformation for $\tau = 0$,

$$X^i(x^j, q^\beta, y^\beta_j; 0) = x^i, \qquad Q^\alpha(x^j, q^\beta, y^\beta_j; 0) = q^\alpha$$
$$Y^\alpha_i(x^j, q^\beta, y^\beta_j; 0) = y^\alpha_i \tag{6.2}$$

and satisfy the invertibility condition

$$\frac{\partial(X^i, Q^\alpha, Y^\alpha_i)}{\partial(x^j, q^\beta, y^\beta_j)} \neq 0 \tag{6.3}$$

for all values of $\tau \in \mathbb{R}$ in a neighborhood of $\tau = 0$ for each point in an open subset of K. These relations are most easily envisioned in terms of graphs on the Lie manifold $L = K \times \mathbb{R}$ of K. The analysis given in Section 2 shows that the one-parameter family of transformations $S(\tau)$ is a family of extended canonical transformations with source horizontal ideal $\mathscr{H}[A_{ij}^{\alpha}] \in \mathfrak{H}(K)$ if and only if the functions $Y_i^{\alpha}(x^j, q^{\beta}, y_j^{\beta}; \tau)$ satisfy the relations

$$Y_k^{\alpha} V_i\langle X^k\rangle = V_i\langle Q^{\alpha}\rangle$$

in which case the $'A$'s are determined by (2.10).

The major difficulty in constructing such a one-parameter family of maps is that of securing satisfaction of the local invertibility condition (6.3). If the one-parameter family $S(\tau)$ were to form a one-parameter Lie pseudogroup, then we could guarantee satisfaction of (6.3) in a neighborhood of $\tau = 0$ for each point in K by integrating the orbital equations of the generating vector field of the corresponding one-dimensional Lie subalgebra. Since we know that extended canonical transformations do not form Lie pseudogroups except under very special circumstances, a different but similar avenue of approach will have to be used.

We can certainly differentiate the presentation relations (6.1) with respect to the parameter τ and then use $S(\tau)^{-1}$ to express the results in terms of the current coordinates $\{'x^i, 'q^q, 'y_i^{\alpha}\}$. This will lead to relations of the form

$$\frac{\partial' x^i}{\partial \tau} = u^i('x^j, 'q^{\beta}, 'y_j^{\beta}; \tau), \qquad \frac{\partial' q^{\alpha}}{\partial \tau} = u^{\alpha}('x^j, 'q^{\beta}, 'y_j^{\beta}; \tau)$$
$$\frac{\partial' y_i^{\alpha}}{\partial \tau} = u_i^{\alpha}('x^j, 'q^{\beta}, 'y_j^{\beta}; \tau) \tag{6.4}$$

which can be integrated subject to the initial data (6.2) to give the presentation relations (6.1) that will necessarily satisfy (6.3) for all τ in a neighborhood of $\tau = 0$. We are therefore naturally led to consider the one-parameter family of vector fields

$$U = u^i('x^j, 'q^{\beta}, 'y_j^{\beta}; \tau)'\partial_i + u^{\alpha}('x^j, 'q^{\beta}, 'y_j^{\beta}; \tau)'\partial_{\alpha} + u_i^{\alpha}('x^j, 'q^{\beta}, 'y_j^{\beta}; \tau)'\partial_{\alpha}^i \tag{6.5}$$

For each numerical value of τ, (6.5) serves to define a vector field on the contact manifold $'K$ with local coordinates $\{'x^i, 'q^{\alpha}, 'y_i^{\alpha}\}$, and hence U defines a vector field on the Lie manifold $L = K \times \mathbb{R}$. The geometric structure defined by (6.5) is thus clear. What now remains is to determine whether we can place restrictions on the choices of the functions $\{u^i, u^{\alpha}, u_i^{\alpha}\}$ so that the vector field U will generate a one-parameter family of extended canonical transformations.

Let $\mathscr{H}[A^\alpha_{ij}]$ be a completely integrable horizontal ideal of $\Lambda(K)$, which we will take to be the source horizontal ideal for the transformations to be considered. We will assume that $S(\tau)$ is an extended canonical transformation with source $\mathscr{H}[A^\alpha_{ij}]$ for some fixed value of τ. Since extended canonical transformations form a group under target-source composition, $S(\tau+\varepsilon)$ should result from the target-source composition of $S(\varepsilon)$ with $S(\tau)$. Thus, if $S(\tau+\varepsilon)$ is a map from K to $''K$ with local coordinates $\{''x^i, ''q^\alpha, ''y^\alpha_i\}$, then the vector field U will generate a one-parameter family of extended canonical transformations only if

$$\begin{gathered} ''x^i = 'x^i + \varepsilon u^i('x^j, 'q^\beta, 'y^\beta_j; \tau) + o(\varepsilon) \\ ''q^\alpha = 'q^\alpha + \varepsilon u^\alpha + o(\varepsilon), \qquad ''y^\alpha_i = 'y^\alpha_i + \varepsilon u^\alpha_i + o(\varepsilon) \end{gathered} \tag{6.6}$$

is an extended canonical transformation with source horizontal ideal $'\mathscr{H}['A^\alpha_{ij}]$ for all values of ε in a sufficiently small neighborhood of $\varepsilon = 0$. Let

$$'V_i = '\partial_i + 'y^\alpha_i\, '\partial_\alpha + 'A^\alpha_{ij}('x^k, 'q^\beta, 'y^\beta_j; \tau)'\partial^j_\alpha, \qquad 1 \le i \le n \tag{6.7}$$

be the canonical basis for $'\mathscr{H}^*['A^\alpha_{ij}]$. The analysis given in Section 2 tells us that the transformation (6.6) will be an extended canonical transformation with source horizontal ideal $'\mathscr{H}['A^\alpha_{ij}]$ if and only if

$$''y^\alpha_k\, 'V_i\langle ''x^k\rangle = 'V_i\langle ''q^\alpha\rangle \tag{6.8}$$

in which case

$$*'A^\alpha_{km}\, 'V_i\langle ''x^k\rangle\, 'V_j\langle ''x^m\rangle = 'V_i\, 'V_j\langle ''q^\alpha\rangle - ''y^\alpha_k\, 'V_i\, 'V_j\langle ''x^k\rangle \tag{6.9}$$

with

$$*'A^\alpha_{ij} = S(\varepsilon)\, *''A^\alpha_{ij} \tag{6.10}$$

It is sufficient to our present purposes to restrict consideration to values of ε sufficiently small that we may neglect all terms of $o(\varepsilon)$. We will therefore use the equivalence relation $a \approx b \Leftrightarrow a = b + o(\varepsilon)$. When the transformation relations (6.6) are substituted into (6.8), we obtain the conditions

$$('y^\alpha_k + \varepsilon u^\alpha_k)(\delta^k_i + \varepsilon\, 'V_i\langle u^k\rangle) \approx 'y^\alpha_i + \varepsilon\, 'V_i\langle u^\alpha\rangle$$

and hence the functions $\{u^\alpha_i('x^j, 'q^\beta, 'y^\beta_j; \tau)\}$ must have the evaluations

$$u^\alpha_i = 'V_i\langle u^\alpha\rangle - 'y^\alpha_k\, 'V_i\langle u^k\rangle \tag{6.11}$$

Since the $'V$'s involve the $'A$'s by (6.7), we will also need equations for the determination of the $'A$'s. In order to obtain these, we substitute the transformation relations (6.6) into (6.9). This yields

$$\begin{aligned} *'A^\alpha_{ij} - 'A^\alpha_{ij} \approx \varepsilon\{ &'V_i\, 'V_j\langle u^\alpha\rangle - 'y^\alpha_k\, 'V_i\, 'V_j\langle u^k\rangle \\ &- *'A^\alpha_{kj}\, 'V_i\langle u^k\rangle - *'A^\alpha_{ki}\, 'V_j\langle u^k\rangle\} \end{aligned} \tag{6.12}$$

Now, (6.10) gives

$$*'A^\alpha_{ij} = S(\varepsilon)\, *''A^\alpha_{ij} \approx ''A^\alpha_{ij} + \varepsilon U\langle ''A^\alpha_{ij}\rangle \tag{6.13}$$

and hence the substitution

$$''A^\alpha_{ij} = 'A^\alpha_{ij} + \varepsilon a^\alpha_{ij} + o(\varepsilon) \tag{6.14}$$

and (6.12) yield

$$\begin{aligned} a^\alpha_{ij} &= 'V_i\, 'V_j\langle u^\alpha\rangle - 'y^\alpha_k\, 'V_i\, 'V_j\langle u^k\rangle - U\langle 'A^\alpha_{ij}\rangle \\ &\quad - 'V^\alpha_{ki}\, 'V_j\langle u^k\rangle - 'A^\alpha_{kj}\, 'V_i\langle u^k\rangle \end{aligned} \tag{6.15}$$

It is now only necessary to note that

$$f(m;\, \tau+\varepsilon) = f(m;\, \tau) + \varepsilon\gamma(m;\, \tau) + o(\varepsilon)$$

implies $\partial f/\partial\tau = \gamma(m;\, \tau)$ in order to glean the statement of the following theorem.

Theorem 6.1. A smooth vector field

$$\begin{aligned} U &= u^i('x^j, 'q^\beta, 'y^\beta_j;\, \tau)'\partial_i + u^\alpha('x^j, 'q^\beta, 'y^\beta_j;\, \tau)'\partial_\alpha \\ &\quad + u^\alpha_i('x^j, 'q^\beta, 'y^\beta_j;\, \tau)'\partial^i_\alpha \end{aligned} \tag{6.16}$$

generates a one-parameter family of extended canonical transformations with source $\mathscr{H}[A^\alpha_{ij}]$ if and only if

$$u^\alpha_i = 'V_i\langle u^\alpha\rangle - 'y^\alpha_k\, 'V_i\langle u^k\rangle \tag{6.17}$$

where

$$'V_i = '\partial_i + 'y^\alpha_i\, '\partial_\alpha + 'A^\alpha_{ij}('x^k, 'q^\beta, 'y^\beta_k;\, \tau)'\partial^j_\alpha \tag{6.18}$$

and the $'A$'s satisfy the evolution equations

$$\begin{aligned} \frac{\partial 'A^\alpha_{ij}}{\partial\tau} &= 'V_i\, 'V_j\langle u^\alpha\rangle - 'y^\alpha_k\, 'V_i\, 'V_j\langle u^k\rangle - U\langle 'A^\alpha_{ij}\rangle \\ &\quad - 'A^\alpha_{ki}\, 'V_j\langle u^k\rangle - 'A^\alpha_{kj}\, 'V_i\langle u^k\rangle \end{aligned} \tag{6.19}$$

subject to the initial data

$$'A^\alpha_{ij}('x^k, 'q^\beta, 'y^\beta_k;\, 0) = A^\alpha_{ij}('x^k, 'q^\beta, 'y^\beta_j) \tag{6.20}$$

When these conditions are met, the finite transformations

$$\begin{gathered} 'x^i = X^i(x^j, q^\beta, y^\beta_j;\, \tau), \qquad 'q^\alpha = Q^\alpha(x^j, q^\beta, y^\beta_j;\, \tau) \\ 'y^\alpha_i = Y^\alpha_i(x^j, q^\beta, y^\beta_j;\, \tau) \end{gathered} \tag{6.21}$$

of the one-parameter family $S(\tau)$ are obtained by solving the system of first-order PDE

$$\frac{\partial X^i}{\partial \tau} = u^i(X^j, Q^\beta, Y_j^\beta; \tau), \qquad \frac{\partial Q^\alpha}{\partial \tau} = u^\alpha(X^j, Q^\beta, Y_j^\beta; \tau)$$
$$\frac{\partial Y_i^\alpha}{\partial \tau} = {}'V_i\langle u^\alpha\rangle - Y_k^\alpha \, {}'V_i\langle u^k\rangle \tag{6.22}$$

subject to the initial data

$$X^i(x^j, q^\beta, y_j^\beta; 0) = x^i, \qquad Q^\alpha(x^j, q^\beta, y_j^\beta; 0) = q^\alpha$$
$$Y_i^\alpha(x^j, q^\beta, y_j^\beta; 0) = y_i^\alpha \tag{6.23}$$

Proof. The previous discussion has shown that a smooth vector field $U \in T(L)$ can generate a one-parameter family of extended canonical transformations only if U has the form given by (6.16) and the $'A$'s satisfy the equations of evolution (6.19) subject to the initial data (6.20). In view of the smoothness of the functions $\{u^i, u^\alpha\}$, standard existence theorems show that the evolution equations (6.19) have solutions in a neighborhood N_1 of $\tau = 0$ that satisfy the initial data (6.20). Once the functions $'A_{ij}^\alpha('x^k, 'q^\beta, 'y_k^\beta; \tau)$ have been determined in this manner, the vector fields $'V_i$ are determined by (6.18), and hence the right-hand sides of the system of equations (6.22) are known functions of the arguments $\{'x^k, 'q^\beta, 'y_k^\beta, \tau\}$. The orbital equations (6.22) of the vector field U therefore become explicitly determined systems of first-order ODE. This system, subject to the initial data (6.23), has a unique solution of the form (6.21) on a neighborhood $N_2 \subset N_1$ of $\tau = 0$, by the standard existence and uniqueness theorem for systems of ODE. The invertibility condition (6.3) is therefore satisfied for all τ in a neighborhood $N_3 \subset N_2$ of $\tau = 0$ by a standard continuity argument based on the fact that the Jacobian matrix of $S(\tau)$ has the value 1 at $\tau = 0$. It thus remains to show that the transformation $S(\tau)$ defined by (6.21) is an extended canonical transformation for all τ in a neighborhood of $\tau = 0$. We know, however, that $S(\tau)$ is an extended canonical transformation for all $\tau \in N_3$ if and only if the equations

$$V_i\langle Q^\beta\rangle = V_i\langle X^k\rangle Y_k^\beta \tag{6.24}$$

$$V_i\langle Y_m^\beta\rangle = V_i\langle X^k\rangle^* A_{km}^\beta \tag{6.25}$$

are satisfied for all $\tau \in N_3$ because these equations imply and are implied by the conditions (2.5) and (2.7), which are both necessary and sufficient for $S(\tau)$ to be an extended canonical transformation. We therefore introduce the quantities

$$M_i^\beta = V_i\langle Q^\beta\rangle - Y_k^\beta V_i\langle X^k\rangle \tag{6.26}$$

$$N_{im}^\beta = V_i\langle Y_m^\beta\rangle - {}'A_{km}^\beta V_i\langle X^k\rangle \tag{6.27}$$

Considered as functions of the parameter τ, the quantities defined by (6.26) and (6.27) satisfy the initial conditions

$$M_i^\beta(0)=0, \qquad N_{im}^\beta(0)=0 \tag{6.28}$$

The result will therefore be established if we can show that

$$M_i^\beta(\tau)=0, \qquad N_{im}^\beta(\tau)=0 \tag{6.29}$$

are valid for all τ in some neighborhood $N_4 \subset N_3$ of $\tau=0$. For any map $S(\tau)$ given by (6.21), the relations (6.26), (6.27), and the chain rule give

$$V_i\langle 'f \circ S(\tau)\rangle = \left\{ V_i\langle X^k\rangle\, 'V_k + M_i^\beta \frac{\partial}{\partial Q^\beta} + N_{im}^\beta \frac{\partial}{\partial Y_m^\beta}\right\}\langle 'f\rangle \tag{6.30}$$

for all $'f \in \Lambda^0('K)$, where

$$'V_k = '\partial k + Y_k^\beta\, '\partial_\beta + 'A_{km}^\beta\, '\partial_k^m$$

As indicated by the notation in (6.30), the transformation equations for $S(\tau)$ are explicitly used in order to express all quantities in terms of the same coordinate cover. Noting that $d/d\tau$ and V_i commute because the vector fields $\{V_i\}$ are defined on K and are thus independent of τ, (6.26) and (6.22) give

$$\begin{aligned}\frac{dM_i^\beta}{d\tau} &= V_i\left\langle \frac{\partial Q^\beta}{\partial \tau}\right\rangle - Y_k^\beta V_i\left\langle \frac{\partial X^k}{\partial \tau}\right\rangle - V_i\langle X^k\rangle \frac{\partial Y_k^\beta}{\partial \tau} \\ &= V_i\langle u^\beta\rangle - Y_k^\beta V_i\langle u^k\rangle - V_i\langle X^k\rangle('V_k\langle u^\beta\rangle - Y_m^\beta\, 'V_k\langle u^m\rangle)\end{aligned} \tag{6.31}$$

Thus, when (6.30) is used to evaluate $V_r\langle\{u^k, u^\beta\}\rangle$, we obtain

$$\frac{dM_i^\beta}{d\tau} = M_i^\gamma\left\{\frac{\partial u^\beta}{\partial Q^\gamma} - Y_m^\beta \frac{\partial u^m}{\partial Q^\gamma}\right\} + N_{ir}^\gamma\left\{\frac{\partial u^\beta}{\partial Y_r^\gamma} - Y_m^\beta \frac{\partial u^m}{\partial Y_r^\gamma}\right\} \tag{6.32}$$

An identical argument starting with N_{im}^β gives

$$\frac{dN_{im}^\beta}{d\tau} = M_i^\gamma\left\{\frac{\partial u_m^\beta}{\partial Q^\gamma} - 'A_{km}^\beta \frac{\partial u^k}{\partial Q^\gamma}\right\} + J_{im}^\beta + N_{ir}^\gamma\left\{\frac{\partial u_m^\beta}{\partial Y_r^\gamma} - 'A_{km}^\beta \frac{\partial u^k}{\partial Y_r^\gamma}\right\} \tag{6.33}$$

where

$$\begin{aligned}J_{im}^\beta = V_i\langle X^r\rangle\Big\{ &'V_r\, 'V_m\langle u^\beta\rangle - Y_k^\beta\, 'V_r\, 'V_m\langle u^k\rangle \\ &- 'A_{km}^\beta\, 'V_r\langle u^k\rangle - 'A_{rk}^\beta\, 'V_m\langle u^k\rangle - \frac{d'A_{rm}^\beta}{d\tau}\Big\}\end{aligned} \tag{6.34}$$

However, the $'A$'s are functions of the current coordinates $\{X^i, Q^\beta, R_i^\beta\}$ and τ, and hence

$$\frac{d'A_{rm}^\beta}{d\tau} = \frac{\partial A_{rm}^\beta}{d\tau} + U\langle 'A_{rm}^\beta\rangle \tag{6.35}$$

Thus, when the evolution equation (6.19) is used to evaluate $\partial' A^{\beta}_{rm}/\partial\tau$ in (6.35) and the results are substituted into the right-hand side of (6.34), we obtain

$$J^{\beta}_{im}=0 \tag{6.36}$$

The system of equations (6.32), (6.33), and (6.36) shows that the quantities $\{M^{\beta}_i(\tau),\ N^{\beta}_{im}(\tau)\}$ satisfy a system of homogeneous, linear, first-order differential equations with smooth coefficients, subject to the homogeneous initial data (6.28). The fundamental existence and uniqueness theorem for such systems shows that there exists a neighborhood N_4 of $\tau=0$ such that

$$M^{\beta}_i(\tau)=0, \qquad N^{\beta}_{im}(\tau)=0 \qquad \forall\tau\in N_4 \tag{6.37}$$

and the result is established. ■

There is an important decomposition of any vector field that generates a one-parameter family of extended canonical transformations with source $\mathscr{H}[A^{\alpha}_{ij}]$. Any such vector field has the form

$$U=u^i\,'\partial_i+u^{\alpha}\,'\partial_{\alpha}+('V_j\langle u^{\alpha}\rangle-'y^{\alpha}_k\,'V_j\langle u^k\rangle)'\partial^j_{\alpha} \tag{6.38}$$

by Theorem 6.1. Now,

$$'V_i='\partial_i+'y^{\alpha}_i\,'\partial_{\alpha}+'A^{\alpha}_{ij}\,'\partial^j_{\alpha}$$

and hence (6.38) can be written in the equivalent form

$$U=u^i\,'V_i+(u^{\alpha}-'y^{\alpha}_k u^k)'\partial_{\alpha}+'V_j\langle u^{\alpha}-'y^{\alpha}_k u^k\rangle'\partial^j_{\alpha} \tag{6.39}$$

because

$$'V_j\langle u^{\alpha}-'y^{\alpha}_k u^k\rangle='V_j\langle u^{\alpha}\rangle-'A^{\alpha}_{jk}u^k-'y^{\alpha}_k\,'V_j\langle u^k\rangle$$

These calculations establish the following result.

Theorem 6.2. Any vector field $U\in T(L)$ that generates a one-parameter family of extended canonical transformations with source $\mathscr{H}[A^{\alpha}_{ij}]\in\mathfrak{H}(K)$ admits the decomposition

$$U=u^i\,'V_i+P \tag{6.40}$$

with

$$P=p^{\alpha}\,'\partial_{\alpha}+'V_i\langle p^{\alpha}\rangle'\partial^i_{\alpha} \tag{6.41}$$

$$u^i=u^i('x^j,'q^{\beta},'y^{\beta}_j;\tau), \qquad p^{\alpha}=p^{\alpha}('x^j,'q^{\beta},'y^{\beta}_j;\tau)$$

and the flow generated by the vector field P leaves the base manifold D_n of the independent variables invariant. In addition, the evolution equations (6.19) that the $'A$'s must satisfy reduce to

$$\frac{\partial' A^{\alpha}_{ij}}{\partial\tau}='V_i\,'V_j\langle p^{\alpha}\rangle-P\langle 'A^{\alpha}_{ij}\rangle \tag{6.42}$$

The collection of all generating vector fields of one-parameter families of canonical transformations with source $\mathscr{H}[A^\alpha_{ij}]$ will be denoted by ect$[A^\alpha_{ij}]$. Let U_1 and U_2 be two elements of ect$[A^\alpha_{ij}]$ and set $'Z_i = '\partial_i + 'r^\alpha_i\, '\partial_\alpha$. Then

$$'V_{(1)i} = 'Z_i + 'A^\alpha_{(1)ij}('x^k, 'q^\beta, 'y^\beta_k; \tau)'\partial^j_\alpha$$

and

$$'V_{(2)i} = 'Z_i + 'A^\alpha_{(2)ij}('x^k, 'q^\beta, 'y^\beta_k; \tau)'\partial^j_\alpha$$

will be different because $'A^\alpha_{(1)ij}$ and $'A^\alpha_{(2)ij}$ satisfy different systems of evolution equations in general. Accordingly, $aU_1 + bU_2$ will not belong to ect$[A^\alpha_{ij}]$ in general because the $'V$'s for $aU_1 + bU_2$ will not be a linear combination of the $'V$'s for U_1 and U_2 in general.

Theorem 6.3. The collection ect$[A^\alpha_{ij}]$ of all generating vector fields of one-parameter families of extended canonical transformations with source $\mathscr{H}[A^\alpha_{ij}]$ does not form a linear subspace of $T(L)$.

There are, however, subsets of ect$[A^\alpha_{ij}]$ that are both linear subspaces and Lie subalgebras of $T(L)$, as the following result shows.

Theorem 6.4. The collection ect$[A^\alpha_{ij}]$ contains the infinite-dimensional Lie algebras ISO$[A^\alpha_{ij}]$ and $\mathbf{pr}^{(1)}(T('G))$.

Proof. It was shown in Section 10 of Edelen (1990) than any isovector of the completely integrable horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ was of the form $U = u^i V_i + W$ with

$$W = \eta^\alpha \partial_\alpha + V_i\langle \eta^\alpha\rangle \partial^i_\alpha$$

for any $\{\eta^\alpha\} \in \Lambda^0(K)$ that satisfy

$$V_i V_j\langle \eta^\alpha\rangle = W\langle A^\alpha_{ij}\rangle$$

Hence, any isovector of $'\mathscr{H}['A^\alpha_{ij}]$ is of the form $U = u^i\, 'V_i + W$ with

$$W = \eta^\alpha\, '\partial_\alpha + 'V_i\langle \eta^\alpha\rangle '\partial^i_\alpha, \qquad \eta^\alpha = \eta^\alpha('x^j, 'q^\beta, 'y^\beta_j)$$

and $\{\eta^\alpha\}$ satisfying

$$'V_i\, 'V_j\langle \eta^\alpha\rangle = W\langle 'A^\alpha_{ij}\rangle$$

Theorem 6.2 shows that any $U = u^i\, 'V_i + W$ belongs to ect$[A^\alpha_{ij}]$, and (6.41) shows that the $'A$'s will satisfy the evolution equations

$$\frac{\partial' A^\alpha_{ij}}{\partial \tau} = 0$$

Since the initial conditions for these equations are

$$'A^\alpha_{ij}('x^k, 'q^\beta, 'y^\beta_j; 0) = A^\alpha_{ij}('x^j, 'q^\beta, 'y^\beta_j)$$

we have

$$'A^{\alpha}_{ij}('x^j, 'q^\beta, 'y^\beta_j; \tau) = A^{\alpha}_{ij}('x^j, 'q^\beta, 'y^\beta_j)$$

that is, the $'A$'s are invariants of this one-parameter family of transformations. Since the $'A$'s do not evolve for any element of ISO[A^{α}_{ij}], the $'V$'s for any two elements of ISO[A^{α}_{ij}] will be the same. Accordingly, ISO[A^{α}_{ij}] forms a linear subspace of $T(L)$, and the results established in Section 11 of Edelen (1990) show that ISO[A^{α}_{ij}] forms a Lie algebra. An element of $\mathbf{pr}^{(1)}(T('G))$ has the form

$$U = u^i\, '\partial_i + u^\alpha\, '\partial_\alpha + ('V_i\langle u^\alpha\rangle - 'y^\alpha_k\, 'V_i\langle u^k\rangle)'\partial^i_\alpha$$

with $u^i = u^i('x^j, 'q^\alpha)$, $u^\alpha = u^\alpha('x^j, 'q^\alpha)$, and hence any element of $\mathbf{pr}^{(1)}(T('G))$ belongs to ect[A^{α}_{ij}]. Since the functions $\{u^i, u^\alpha\}$ do not depend on the arguments $\{'y^\beta_i\}$, we have

$$'V_i\langle\{u^i, u^\alpha\}\rangle = 'Z_j\langle\{u^i, u^\alpha\}\rangle$$

with

$$'Z_i = '\partial_i + 'y^\alpha_i\, '\partial_\alpha$$

Hence, any $U \in \mathbf{pr}^{(1)}(T('G))$ takes the equivalent form

$$U = u^i\, '\partial_i + u^\alpha\, '\partial_\alpha + 'Z_i\langle u^\alpha - 'y^\alpha_k u^k\rangle'\partial^i_\alpha$$

Thus, since the $'Z$'s are the same for all elements of $\mathbf{pr}^{(1)}(T('G))$, linear combinations of elements of $\mathbf{pr}^{(1)}(T('G))$ are elements of ect[A^{α}_{ij}] even though the $'A$'s are τ-dependent because they satisfy the evolution equations (6.19). Further, we know that $\mathbf{pr}^{(1)}(T('G))$ forms a Lie algebra that is universal with respect to the choice of the source horizontal ideal and that $'K$ and K coincide for $\tau = 0$. We have therefore shown that the Lie algebras ISO[A^{α}_{ij}] and $\mathbf{pr}^{(1)}(T('G))$ are contained in ect[A^{α}_{ij}]. ■

There is much more here than first meets the eye, however. Let ISO[A^{α}_{ij}]: $\Lambda^0(\mathbb{R})$ and $\mathbf{pr}^{(1)}(T('G))$: $\Lambda^0(\mathbb{R})$ be the modules constructed from ISO[A^{α}_{ij}] and $\mathbf{pr}^{(1)}(T('G))$ where multiplication by elements of $\mathbb{R}$ is replaced by multiplication by smooth functions of τ. An element W of $\mathbf{pr}^{(1)}(T('G))$: $\Lambda^0(\mathbb{R})$ thus looks like

$$W = u^i('x^k, 'q^\beta; \tau)'\partial_i + u^\alpha('x^k, 'q^\beta; \tau)'\partial_\alpha + 'Z_i\langle u^\alpha - 'y^\alpha_k u^k\rangle'\partial^i_\alpha$$

Calculations similar to those given in the proof of Theorem 6.4 show that *the modules* ISO[A^{α}_{ij}]: $\Lambda^0(\mathbb{R})$ *and* $\mathbf{pr}^{(1)}(T('G))$: $\Lambda^0(\mathbb{R})$ *are Lie subalgebras of* $T(L)$ *that are contained in* ect[A^{α}_{ij}]. These Lie algebras are not the Lie algebras of Lie groups, in general. In order to see this, consider the Lie subalgebra of $\mathbf{pr}^{(1)}(T('G))$ that is generated by the r elements $\{U_1, U_2, \ldots, U_r\}$; that is, $[\![U_a, U_b]\!] = C^c_{ab} U_c$. The vector fields $\{W_a = N^b_a(\tau) U_b | 1 \le a \le r,\ \det(N^b_a(\tau)) \neq 0\}$ will generate a Lie subalgebra of

$\mathbf{pr}^{(1)}(T('G))$: $\Lambda^0(\mathbb{R})$ with

$$[\![W_a, W_b]\!] = K^e_{ab}(\tau)\, W_e$$

$$K^e_{ab}(\tau) = N^s_a(\tau) N^t_b(\tau) C^c_{ab} n^e_c(\tau), \qquad N^a_b(\tau) n^b_c(\tau) = \delta^a_c$$

Since the vector fields $\{W_a | 1 \le a \le r\}$ are vector fields on $'L = 'K \times \mathbb{R}$, we have *functions of structure* $K^u_{ab}(\tau)$ rather than constants of structure. Accordingly, the second fundamental theorem of Lie tells us that the Lie algebra generated by $\{W_a\}$, as an algebra over $\mathbb{R}$, is not the Lie algebra of germs of a Lie group.

We are able to get these more general Lie algebras as algebras over the ring of smooth functions of τ because ECT forms a group under target-source composition and we are certainly able to change from one element of ISO$[A^\alpha_{ij}]$ (of $\mathbf{pr}^{(1)}(T('G))$) to another as we move from one value of τ to another. For example, for $n = 2$, $N = 1$, with local coordinates $\{x, t, q, y_x, y_t\}$ on K, the vector field

$$W = ('q)^2\, e^\tau\, '\partial_q + 2'q\, e^\tau ('y_x\, '\partial^x + 'y_t\, '\partial^t)$$

generates the one-parameter family of extended canonical transformations

$$'x = x, \qquad 't = t, \qquad 'q = \frac{q}{1 + q(1 - e^\tau)}$$

$$'y_x = \frac{y_x}{\{1 + q(1 - e^\tau)\}^2}, \qquad 'y_t = \frac{y_t}{\{1 + q(1 - e^\tau)\}^2}$$

for any completely integrable source horizontal ideal because W belongs to $\mathbf{pr}^{(1)}(T('G))$: $\Lambda^0(\mathbb{R})$ and this module is universal with respect to the choice of source. Thus, for example, if we take $A^1_{ij} = k_{ij} = k_{ji}$, where the k's are constants, then direct calculations based on (2.7) give

$$'A^1_{ij} = \{1 - 'q(1 - e^\tau)\}^2 k_{ij} - \frac{2(1 - e^\tau)'y_i\, 'y_j}{1 - 'q(1 - e^\tau)}$$

It is then easily checked that these evaluations satisfy the evolution equations (6.19) subject to the initial data (6.20).

Theorem 4.4 shows that extended canonical transformations that change the A's can be restricted so that the independent variables are invariants of the transformation. We also know that any $U \in \text{ect}[A^\alpha_{ij}]$ can be written in the form $U = u^i('x^j, 'q^\beta, 'y^\beta_j; \tau)\, 'V_i + P$, that any vector field of the form $u^i\, 'V_i$ belongs to ISO$['A^\alpha_{ij}]$: $\Lambda(\mathbb{R})$, and that $u^i\, 'V_i$ generates automorphisms of the leaves of the foliation generated by $'\mathscr{H}['A^\alpha_{ij}]$ (see Section 12 of Edelen, 1990). It is therefore useful to study the subset

$$\text{ect}_\perp[A^\alpha_{ij}] = \{P = p^\alpha\, '\partial_\alpha + 'V_i\langle p^\alpha\rangle\, '\partial^i_\alpha | p^\alpha \in \Lambda^0(L)\} \tag{6.43}$$

of ect$[A^\alpha_{ij}]$. Vector fields P in ect$_\perp[A^\alpha_{ij}]$ have the property that the evolution equations (6.19) reduce to

$$\frac{\partial' A^\alpha_{ij}}{\partial\tau} = {'V_i}\,{'V_j}\langle p^\alpha\rangle - P\langle' A^\alpha_{ij}\rangle \tag{6.44}$$

In particular, for any

$$P = p^\alpha('x^k, 'q^\beta; \tau)'\partial_\alpha + {'Z_i}\langle p^\alpha\rangle\, '\partial^i_\alpha \tag{6.45}$$

in ect$_\perp[A^\alpha_{ij}] \cap \mathbf{pr}^{(1)}(T('G))$: $\Lambda^0(\mathbb{R})$, the evolution equations (6.44) reduce to the quasilinear system

$$\left(\frac{\partial}{\partial\tau} + P\right)\langle' A^\alpha_{ij}\rangle = {'A^\beta_{ij}}\,'\partial_\beta\langle p^\alpha\rangle + {'Z_i}\,{'Z_j}\langle p^\alpha\rangle \tag{6.46}$$

with the same principal part. They can therefore be solved directly by the method of characteristics. We hasten to remind the reader that the set ect$_\perp[A^\alpha_{ij}]$ does not form a subspace of $T(L)$ and therefore does not form a Lie subalgebra of $T(L)$ because vector fields $P = p^\alpha\, '\partial_\alpha + {'V_i}\langle p^\alpha\rangle\, '\partial^i_\alpha \in$ ect$_\perp[A^\alpha_{ij}]$ with different choices of $\{p^\alpha\}$ will lead to different $'A$'s in general.

7. AUTOBALANCE TRANSFORMATIONS AND THE GENERATION OF SOLUTIONS

Let $\mathscr{H}[A^\alpha_{ij}]$ be a completely integrable horizontal ideal of $\Lambda(K)$ and let $\Psi\colon D_n \to K$ be a solution map of the *balance ideal*

$$\mathscr{B} = I\{C^\alpha, H^\alpha_i, B_a\} \tag{7.1}$$

The map Ψ thus satisfies

$$\Psi^*\mu \neq 0, \qquad \Psi^* C^\alpha = 0, \qquad \Psi^* H^\alpha_i = 0, \qquad \Psi^* B_a = 0 \tag{7.2}$$

and hence Ψ is also a solution map of the fundamental ideal $\mathscr{I} = I\{C^\alpha, dC^\alpha, B_a, dB_a\}$ that satisfies the constraints $\Psi^* H^\alpha_i = 0$. If $P \in$ ect$_\perp[A^\alpha_{ij}]$ generates the one-parameter family of elements $S_P(\tau)$ of ECT, then $S_P(\tau)$ carries the balance ideal $\mathscr{B}$ of $\Lambda(K)$ into the balance ideal

$$'\mathscr{B} = I\{'C^\alpha, 'H^\alpha_i, 'B_a\} \tag{7.3}$$

of $\Lambda('K)$ with balance n-forms

$$'B_a = (S_P(\tau)^{-1})^* B_a \tag{7.4}$$

The one-parameter family of maps

$$\Psi_P(\tau) = S_P(\tau) \circ \Psi \tag{7.5}$$

thus satisfies

$$\Psi_P(\tau)^*\mu = \Psi^* \circ S_P(\tau)^*\mu = \Psi^*\mu \neq 0 \tag{7.6}$$

because any $S_P(\tau)$ leaves the base manifold invariant for $P \in \mathrm{ect}_\perp[A^\alpha_{ij}]$, and

$$\Psi_P(\tau)^{*\prime}\mathscr{B} = \Psi^* \circ S_P(\tau)^{*\prime}\mathscr{B} = \Psi^*\mathscr{B} = 0 \tag{7.7}$$

This shows that the one-parameter family of maps $\Psi_P(\tau): D_n \to {}'K$ solves the balance ideal ${}'\mathscr{B}$. It therefore solves the fundamental ideal

$${}'\mathscr{I} = I\{{}'C^\alpha, d'C^\alpha, {}'B_a, d'B_a\} \tag{7.8}$$

This process of generating solving maps of the target balance ideal ${}'\mathscr{B}$ is not effective for arbitrary generating vectors $P \in \mathrm{ect}_\perp[A^\alpha_{ij}]$ because the target balance n-forms

$${}'B_a = (S_P(\tau)^{-1})^* B_a \tag{7.9}$$

will be different from the balance n-forms B_a that we wish to solve. The structure of the relations (7.9) shows that there can be elements P of $\mathrm{ect}_\perp[A^\alpha_{ij}]$ for which the target balance n-forms in the coordinate cover of ${}'K$ will have an identical functional form to the original balance ideal will be a solving map for the original system of PDE of the problem. These preferred elements of $\mathrm{ect}_\perp[A^\alpha_{ij}]$ are those for which

$$\pounds_P\, {}'B_a \equiv N^b_a\, {}'B_b \bmod \mathscr{H}[A^\alpha_{ij}], \qquad 1 \leq a \leq r \tag{7.10}$$

where we take

$${}'B_a = h_a({}'x^k, {}'q^\beta, {}'y^\beta_k)'\mu - dW^i_a({}'x^k, {}'q^\beta, {}'y^\beta_k) \wedge \mu_i$$

when the original balance n-forms on K are given by

$$B_a = h_a(x^k, q^\beta, y^\beta_k)\mu - dW^i_a(x^k, q^\beta, y^\beta_j) \wedge \mu_i$$

Definition 7.1. An element P of $\mathrm{ect}_\perp[A^\alpha_{ij}]$ that satisfies the system of R equations (7.10) is referred to as an *autobalance* vector field of the balance ideal $\mathscr{B}$. The collection of all autobalance vectors of $\mathscr{B}$ is denoted by

$$\mathrm{aut}_\perp[\mathscr{B}] = \{P \in \mathrm{ect}_\perp[A^\alpha_{ij}] | \pounds_P\, {}'B_a \equiv N^b_a\, {}'B_b \bmod {}'\mathscr{H}[{}'A^\alpha_{ij}]\} \tag{7.11}$$

These considerations have established the following basic result.

Theorem 7.1. Let $\mathscr{H}[A^\alpha_{ij}]$ be a completely integrable horizontal ideal of $\Lambda(K)$ and let $\Psi: D_n \to K$ be a leaf map of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ that is also a solution map of the balance ideal $\mathscr{B} = I\{C^\alpha, H^\alpha_i, B_a\}$ for a system of PDE with balance n-forms B_a. Each element of the one-parameter family of maps

$$\Psi_P(\tau) = S_P(\tau) \circ \Psi \tag{7.12}$$

is a solution map of the fundamental ideal for each autobalance vector P of the balance ideal $\mathscr{B}$.

What now remains is to obtain explicit characterization of the conditions under which an element P of $\text{ect}_\perp[A^\alpha_{ij}]$ will be an autobalance vector of $\mathscr{B}$.

Theorem 7.2. Let $\mathscr{H}[A^\alpha_{ij}]$ be a completely integrable horizontal ideal of $\Lambda(K)$ and let $\{V_i|1\le i\le n\}$ be the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$. A vector field $P\in T(L)$ is an autobalance vector of the balance ideal $'\mathscr{B}=I\{'C^\alpha, 'H^\alpha_i, 'B_a\}$ with balance n-forms

$$'B_a=h_a('x^k, 'q^\beta, 'y^\beta_k)'\mu - dW^i_a('x^k, 'q^\beta, 'y^\beta_k)\wedge '\mu_i \tag{7.13}$$

if and only if P has the form

$$P=p^\alpha\,'\partial_\alpha + 'V_i\langle p^\alpha\rangle '\partial^i_\alpha \tag{7.14}$$

and the N generating functions $\{p^\alpha\in\Lambda^0('K_1)\}$ satisfy the system of r linear, second-order PDE

$$P\langle h_a\rangle - 'V_iP\langle W^i_a\rangle = N^b_a(h_b - 'V_i\langle W^i_b\rangle), \qquad 1\le a\le r \tag{7.15}$$

for some choice of the functions $\{N^b_a\in\Lambda^0(L)\}$.

Proof. A vector field $P\in T(K)$ belongs to $\text{ect}_\perp[A^\alpha_{ij}]$ if and only if it is of the form (7.14). Therefore, since $\text{aut}_\perp[\mathscr{B}]\subset \text{ect}_\perp[A^\alpha_{ij}]$, we may assume that P is given by (7.14) for some choice of the N generating functions $\{p^\alpha\}$. Since $P\,\lrcorner\, d'x^i=0$, it follows that $\pounds_P\,'\mu=0$ and $\pounds_P\,'\mu_i=0$, and hence (7.13) yield the relations

$$\pounds_P\,'B_a = P\langle h_a\rangle '\mu - d(P\langle W^i_a\rangle)\wedge '\mu_i \tag{7.16}$$

However, $df='V_j\langle f\rangle\, d'x^j \bmod '\mathscr{H}['A^\alpha_{ij}]$, $d'x^j\wedge '\mu_i=\delta^j_i\,'\mu$, and hence

$$\pounds_P\,'B_a \equiv \{P\langle h_a\rangle - 'V_iP\langle W^i_a\rangle\}\,'\mu \bmod '\mathscr{H}['A^\alpha_{ij}] \tag{7.17}$$

It thus follows that

$$\pounds_P\,'B_a\equiv N^b_aB_b \bmod '\mathscr{H}['A^\alpha_{ij}] \equiv N^b_a(h_b - 'V_i\langle W^i_b\rangle\,'\mu \bmod '\mathscr{H}['A^\alpha_{ij}]$$

if and only if the N generating functions $\{p^\alpha\}$ satisfy the system of r second-order PDE given by (7.15). ■

Remark. The reader may wish to pursue the theory with elements

$$U=u^i\,'\partial_i + u^\alpha\,'\partial_\alpha + ('V_i\langle u^\alpha\rangle - 'y^\alpha_k\,'V_i\langle u^k\rangle)'\partial^i_\alpha \tag{7.18}$$

of $\text{ect}[A^\alpha_{ij}]$ in general position. A calculation similar to that given above shows that the conditions (7.15) are replaced by

$$\begin{aligned}&U\langle h_a\rangle + (h_a - 'V_i\langle W^i_a\rangle)\,'V_m\langle u^m\rangle\\ &\quad - 'V_iU\langle W^i_a\rangle + 'V_i\langle u^k\rangle\,'V_k\langle W^i_a\rangle = N^b_a(h_b - 'V_i\langle W^i_b\rangle)\end{aligned} \tag{7.19}$$

A direct combination of Theorems 7.1 and 7.2 provides the following results that are instrumental in obtaining solutions to nonlinear systems of balance equations.

Theorem 7.3. Let $\mathscr{H}[A^\alpha_{ij}]$ be a completely integrable horizontal ideal and let $\Psi: D_n \to K$ be a solution map of the balance ideal $\mathscr{B}$ with balance n-forms

$$B_a = h_a(x^k, q^\beta, y^\beta_k)\mu - dW^i_a(x^k, q^\beta, y^\beta_k) \wedge \mu_i \tag{7.20}$$

Each collection of N generating functions $\{p^\alpha \in \Lambda^0(L_1) | 1 \le \alpha \le N\}$ that satisfy the system of r second-order PDE

$$P\langle \hat{h}_a\rangle - 'V_i P\langle \hat{W}^i_a\rangle = N^b_a(\hat{h}_b - 'V_i\langle \hat{W}^i_b\rangle), \qquad 1 \le a \le r \tag{7.21}$$

where $P = p^\alpha\, '\partial_\alpha + 'V_i\langle p^\alpha\rangle' \partial^i_\alpha$ and

$$\hat{h}_a = h_a('x^k, 'q^\beta, 'y^\beta_k), \qquad \hat{W}^i_a = W^i_a('x^k, 'q^\beta, 'y^\beta_k) \tag{7.22}$$

gives rise to a one-parameter family

$$\Psi_P(\tau) = S_P(\tau) \circ \Psi \tag{7.23}$$

of solution maps of the fundamental ideal.

Autobalance vector fields of a given balance ideal are generalizations of the notion of isovector fields of a fundamental ideal. Elements of $\mathrm{aut}_\perp[\mathscr{B}]$ serve the same purpose as elements of $\mathrm{iso}[\mathscr{I}]$; namely, they implement the embedding of any solution map in a one-parameter family of solution maps. In particular, they give immediate information concerning the connectivity of solution maps of the system of PDE under study. The set $\mathrm{aut}_\perp[\mathscr{B}]$ is intrinsically different from $\mathrm{iso}[\mathscr{I}]$ because $\mathrm{iso}[\mathscr{I}]$ forms a Lie algebra, while $\mathrm{aut}_\perp[\mathscr{B}]$ does not even form a linear subspace of $T('K)$ over $\mathbb{R}$.

The usefulness of these considerations depends critically on the fact that Ψ is a solving map of the balance ideal $\mathscr{B} = I\{C^\alpha, H^\alpha_i, B_a\}$, and hence the graph of Ψ is a leaf of the foliation of K_1 that is generated by the completely integrable horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$. This means that if we know a solution map Φ of the fundamental ideal, then we must find the completely integrable horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ such that $\Phi^* H^\alpha_i = 0$; that is, we have to find the appropriate A's. This is an easy task, however. If Φ has the presentation

$$\Phi | x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^k), \qquad y^\alpha_i = \partial_i \phi^\alpha(x^k)$$

then the choices

$$A^\alpha_{ij} = \partial_i \partial_j \phi^\alpha(x^k) \tag{7.24}$$

give a completely integrable horizontal ideal $\mathcal{H}[\partial_i\partial_j\phi^\alpha]$ such that the graph of Φ is a leaf of the foliation of K that is generated by $\mathcal{H}[\partial_i\partial_j\phi^\alpha]$. Thus, Theorem 7.3 shows how to embed any solution map of the fundamental ideal in one-parameter families of solution maps for each solution of the system (7.21) of r second-order PDE in N unknowns

$$\{p^\alpha \in \Lambda^0(K) | 1 \leq \alpha \leq N\}$$

There is a subtle aspect of this problem that has yet to be faced, however. The vector fields $'V_i$ have the evaluation

$$'V_i = '\partial_i + 'y_i^\alpha\, '\partial_\alpha + 'A_{ij}^\alpha\, '\partial_\alpha^j \tag{7.25}$$

so they depend explicitly on the $'A$'s. The $'A$'s must satisfy the system of evolution equations (6.30), which also depend explicitly on the functions $\{p^\alpha \in \Lambda^0(L)\}$. We therefore have to solve the simultaneous system of partial differential equations

$$P\langle \hat{h}_a\rangle - 'V_i P\langle \hat{W}_a^i\rangle = N_a^b(\hat{h}_b - 'V_i\langle \hat{W}_b^i\rangle) \tag{7.26}$$

$$\frac{\partial ' A_{ij}^\alpha}{\partial \tau} = 'V_i\, 'V_j\langle p^\alpha\rangle - P\langle 'A_{ij}^\alpha\rangle \tag{7.27}$$

subject to the initial data

$$'A_{ij}^\alpha('x^k, 'q^\beta, 'y_k^\beta; 0) = A_{ij}^\alpha('x^k, 'q^\beta, 'y_k^\beta) \tag{7.28}$$

for the functions $p^\alpha('x^k, 'q^\beta, 'y_k^\beta; \tau)$ and $'A_{ij}^\alpha('x^k, 'q^\beta, 'y_k^\beta; \tau)$, where

$$P = p^\alpha\, '\partial_\alpha + 'V_i\langle p^\alpha\rangle '\partial_\alpha^i \tag{7.29}$$

$$'V_i = '\partial_i + 'y_i^\alpha\, '\partial_\alpha + 'A_{ij}^\alpha\, '\partial_\alpha^j \tag{7.30}$$

It is thus through the initial data (7.28) that the information contained in the starting horizontal ideal $\mathcal{H}[A_{ij}^\alpha]$ is included. For example, if the A's are chosen in accord with (7.24), then the initial data (7.28) become

$$'A_{ij}^\alpha('x^k, 'q^\beta, 'y_k^\beta; 0) = \frac{\partial^2\phi('x^k)}{\partial' x^i \partial' x^j}$$

Most of the results given in Edelen (1990) use the notation

$$\hat{F}_a = \hat{h}_a - 'V_i\langle \hat{W}_a^i\rangle \tag{7.31}$$

Solving the system (7.31) for $\{\hat{h}_a\}$, we have

$$P\langle \hat{h}_a\rangle = P\langle \hat{F}_a\rangle + P'V_i\langle \hat{W}_a^i\rangle \tag{7.32}$$

Thus, since $[\![P, 'V_i]\!]\langle \hat{W}_a^i\rangle = P'V_i\langle \hat{W}_a^i\rangle - 'V_i P\langle \hat{W}_a^i\rangle$, equations (7.21) take the equivalent form

$$P\langle \hat{F}_a\rangle + [\![P, 'V_i]\!]\langle \hat{W}_a^i\rangle = N_a^b \hat{F}_b, \qquad 1 \leq a \leq r \tag{7.33}$$

In particular, if $'\mathscr{H}['A^\alpha_{ij}] \in \mathfrak{H}_s('K)$, then $\hat{F}_a = 0$, $1 \le a \le r$, and every leaf of the foliation generated by $'\mathscr{H}['A^\alpha_{ij}]$ is the graph of a solution map of the fundamental ideal. Under these conditions, (7.33) reduce to the drastically simplified, but nontrivial requirements

$$[\![P, 'V_i]\!]\langle \hat{W}^i_a \rangle = 0, \qquad 1 \le a \le r \tag{7.34}$$

These results have a number of aspects in common with those obtained for isovector fields of the fundamental ideal $\mathscr{I}$ of a given system of balance n-forms. Isovector fields of the fundamental ideal will usually induce flows that change the independent variables $\{x^i\}$. We will therefore drop the requirement that only elements of $\mathrm{ect}_\perp[A^\alpha_{ij}]$ be considered. We therefore consider the set

$$\mathrm{aut}[\mathscr{B}] = \{U \in \mathrm{ect}[A^\alpha_{ij}] | \pounds_U\, 'B_a \equiv L^b_a\, 'B_b \bmod '\mathscr{H}['A^\alpha_{ij}]\} \tag{7.35}$$

of generating vector fields of unrestricted autobalance transformations of the balance ideal. The equations that determine elements of $\mathrm{aut}[\mathscr{B}]$ are (7.19).

Theorem 7.4. The collection $\mathrm{aut}[\mathscr{B}]$ contains the Lie algebra $\mathrm{iso}[\mathscr{I}] \cap \mathbf{pr}^{(1)}(T('G))$ of all isovector fields of the fundamental ideal.

Proof. The intersection of Lie algebras given in the hypothesis is to exclude those elements of $\mathrm{iso}[\mathscr{I}]$ that are not prolongations when $N = 1$. Since every element of the indicated Lie algebra is a prolongation, $\mathbf{pr}^{(1)}(T('G)) \subset \mathrm{ect}[A^\alpha_{ij}]$, and prolongations are universal with respect to the choice of the source horizontal ideal, it follows directly from the definition of $\mathrm{iso}[\mathscr{I}]$ that $\mathrm{iso}[\mathscr{I}] \cap \mathbf{pr}^{(1)}(T('G))$ belongs to $\mathrm{aut}[\mathscr{B}]$. ■

This theorem is important for two reasons. First, it shows that any system of equations of balance with a nontrivial isogroup necessarily admits a nontrivial collection of one-parameter families of unrestricted autobalance transformations. Second, since $\mathrm{iso}[\mathscr{I}] \cap \mathbf{pr}^{(1)}(T('G))$ is contained in $\mathrm{aut}[\mathscr{B}]$, the collection $\mathrm{aut}[\mathscr{B}]$ is possibly much larger than $\mathrm{iso}[\mathscr{I}] \cap \mathbf{pr}^{(1)}(T('G))$. In fact, it is clear that those elements of $\mathrm{aut}[\mathscr{B}]$ that do not belong to $\mathrm{iso}[\mathscr{I}] \cap \mathbf{pr}^{(1)}(T('G))$ are vector fields that generate symmetry transformations of the system of equations of balance that *cannot* be obtained in the standard jet bundle formulation. Similar conclusions hold for the collection $\mathrm{aut}_\perp[\mathscr{B}]$.

8. SOLUTIONS OF THE NAVIER–STOKES EQUATIONS

One of the oldest nonlinear problems is that of obtaining solutions to the Navier–Stokes equations for an incompressible fluid. Let $\tilde{\mu}$ be the viscosity, ρ be the density, P be the pressure field, and introduce the parameters $p = P/\rho$, $M = \tilde{\mu}/\rho$. The governing PDE are

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla p = M\nabla^2 \mathbf{U}, \qquad \nabla \cdot \mathbf{U} = 0 \tag{8.1}$$

where $\mathbf{U}$ is the velocity vector field of the fluid. Although the problem can be studied in this general context, in which case the contact manifold is of dimension 24, we restrict the problem in the interests of simplicity to flows with only two spatial dimensions. This means that $\mathbf{U} = u(x, z, t)\mathbf{i} + w(x, z, t)\mathbf{k}$ and that the pressure is a function of the variables $\{x, z, t\}$ only.

The appropriate contact manifold K has dimension 15 with a system of local coordinates

$$\{z^A\} = \{x, z, t, u, w, p, y^u_x, y^u_z, y^u_t, y^w_x, y^w_z, y^w_t, y^p_x, y^p_z, y^p_t\} \tag{8.2}$$

The contact 1-forms that are required for the problem are

$$\begin{aligned} C^u &= du - y^u_x\,dx - y^u_z\,dz - y^u_t\,dt \\ C^w &= dw - y^w_x\,dx - y^w_z\,dz - y^w_t\,dt \\ C^p &= dp - y^p_x\,dx - y^p_z\,dz - y^p_t\,dt \end{aligned} \tag{8.3}$$

The volume element and the boundary elements of the base manifold $D_3 \subset \mathbb{R}^3$ are given by $\mu = dx \wedge dz \wedge dt$, $\mu_x = dz \wedge dt$, $\mu_z = -dx \wedge dt$, $\mu_t = dx \wedge dz$. The field equations (8.1) are then encoded by the balance 3-forms

$$\begin{aligned} B_1 &= (y^u_t + uy^u_x + wy^u_z + y^p_x)\mu - dN_1 \\ B_2 &= (y^w_t + uy^w_x + wy^w_z + y^p_z)\mu - dN_2 \\ B_3 &= (y^u_x + y^w_z)\mu \end{aligned} \tag{8.4}$$

where

$$N_1 = M(y^u_x\mu_x + y^u_z\mu_z), \qquad N_2 = M(y^w_x\mu_x + y^w_z\mu_z) \tag{8.5}$$

These balance 3-forms are easily seen to be of the standard form $B_a = h_a\mu - dW^i_a \wedge \mu_i$. The balance ideal for this problem is therefore of the form $\mathscr{B} = I\{C^\alpha, H^\alpha_i, B_a | 1 \le i,\ \alpha,\ a \le 3\}$. Since there are only two spatial dimensions, the vorticity vector has the representation $\boldsymbol{\eta} = \eta\mathbf{j}$ and η has the representation

$$\eta = y^u_z - y^w_x \tag{8.6}$$

on K.

Let $\mathscr{H}[A^u_{ij}, A^w_{ij}, A^p_{ij}]$ be a completely integrable horizontal ideal of $\Lambda(K)$, and let $\{V_x, V_z, V_t\}$ be the canonical basis for $\mathscr{H}^*[A^u_{ij}, A^w_{ij}, A^p_{ij}]$. We will only write out one of these, in view of their length:

$$\begin{aligned} V_x = \partial_x &+ y^u_x\partial_u + y^w_x\partial_w + y^p_x\partial_p + A^u_{xx}\partial^x_u + A^u_{xz}\partial^z_u + A^u_{xt}\partial^t_u \\ &+ A^w_{xx}\partial^x_w + A^w_{xz}\partial^z_w + A^w_{xt}\partial^t_w + A^p_{xx}\partial^x_p + A^p_{xz}\partial^z_p + A^p_{xt}\partial^t_p \end{aligned} \tag{8.7}$$

A vector field

$$P = p^u\,'\partial_u + p^w\,'\partial_w + p^p\,'\partial_p + 'V_i\langle p^u\rangle'\partial_u^i + 'V_i\langle p^w\rangle'\partial_w^i + 'V_i\langle p^p\rangle'\partial_p^i \tag{8.8}$$

belongs to $\mathrm{aut}_\perp[\mathscr{B}]$ if and only if the generating functions $\{p^u('z^A;\tau)$, $p^w('z^A;\tau)$, $p^p('z^A;\tau)\}$ satisfy the system of second-order PDE

$$P\langle\hat{h}_a\rangle - 'V_i P\langle\hat{W}_a^i\rangle = N_a^b(\hat{h}_b - 'V_i\langle\hat{W}_b^i\rangle) \tag{8.9}$$

For the purposes of this discussion we will only consider elements of $\mathrm{aut}_\perp[\mathscr{B}]$ for which $N_a^b = 0$; that is, autobalance vector fields for which the balance n-forms are absolute invariants. An expansion of (8.9) by use of (8.4) and (8.5) gives the explicit equations

$$\begin{aligned}&'V_t\langle p^u\rangle + p^u\,'y_x^u + p^w\,'y_z^u + ('u'V_x + 'w'V_z)\langle p^u\rangle + 'V_x\langle p^p\rangle\\ &\quad = M('V_x\,'V_x + 'V_z\,'V_z)\langle p^u\rangle\end{aligned} \tag{8.10}$$

$$\begin{aligned}&'V_t\langle p^w\rangle + p^u\,'y_x^w + p^w\,'y_z^w + ('u'V_x + 'w'V_z)\langle p^w\rangle + 'V_z\langle p^p\rangle\\ &\quad = M('V_x\,'V_x + 'V_z\,'V_z)\langle p^w\rangle\end{aligned} \tag{8.11}$$

$$'V_x\langle p^u\rangle + 'V_z\langle p^w\rangle = 0 \tag{8.12}$$

Any solution $\{p^u, p^w, p^p\}$ of these equations will give a one-parameter family $S_P(\tau)$ of extended canonical transformations that maps the source balance ideal $\mathscr{B}$ into the target balance ideal $'\mathscr{B}$ and preserves the functional forms of the balance 1-forms. It must be carefully noted, however, that

$$'V_i = '\partial_i + 'y_i^\alpha\,'\partial_\alpha + 'A_{ij}^\alpha\,'\partial_\alpha^j \tag{8.13}$$

which involve the $'A$'s. Accordingly, the equations (8.10)–(8.12) have to be solved simultaneously with the evolution equations

$$\frac{\partial' A_{ij}^\alpha}{\partial\tau} = 'V_i\,'V_j\langle p^\alpha\rangle - P\langle 'A_{ij}^\alpha\rangle \tag{8.14}$$

subject to the initial data

$$'A_{ij}^\alpha('z^A; 0) = A_{ij}^\alpha('z^A) \tag{8.15}$$

These considerations establish the following result.

If $\Psi : D_3 \to K$ is a leaf map of the completely integrable horizontal ideal $\mathscr{H}[A_{ij}^u, A_{ij}^w, A_{ij}^p]$ that solves the balance ideal for the Navier–Stokes equations of an incompressible, two-dimensional flow, if

$$\{p^u('z^A;\tau), p^w('z^A;\tau), p^p('z^A;\tau), 'A_{ij}^u('z^A;\tau), 'A_{ij}^w('z^A;\tau), 'A_{ij}^p('z^A;\tau)\}$$

is any solution of the second-order system of PDE (8.10)–(8.12), and the evolution equations (8.14) subject to the initial data (8.15), if the vector field P is defined by (8.8), and if $S_P(\tau)$ is the one-parameter family of extended canonical transformations generated by P, then the one-parameter family of maps $\Psi_P(\tau) = S_P(\tau)\circ\Psi$ solves the fundamental ideal of the Navier–Stokes equations for each value of τ in a neighborhood of $\tau = 0$.

The reader should note that each of the maps $\Psi_P(\tau)$ has a common domain, J_3CD_3, because the flow of the vector field P leaves the base manifold D_3 invariant.

Obvious starting solutions for problems with the Navier–Stokes equations are those for potential flow of a perfect fluid. These solutions are irrotational and give rise to zero viscous forces. Let $\{p^u, p^w, p^p\}$ be any solution of the system (8.10)–(8.12) for a source ideal for which one leaf of its foliation is a solution of the Navier–Stokes equations for a potential flow. The one-parameter family of solutions that embeds the starting solution will have nonzero viscous forces, in general. If τ is the parameter of the embedding, then to first-order terms in τ, (8.6) gives

$$'\eta \approx \eta + \tau P\langle y_z^u - y_x^w\rangle = \eta + \tau(V_z\langle p^u\rangle - V_x\langle p^w\rangle) \tag{8.16}$$

The resulting solutions for $\tau \neq 0$ will thus have nonzero vorticity, in general. A similar evaluation of the viscous forces gives

$$\begin{aligned} d'N_1 &\approx dN_1 + \tau(V_xV_x + V_zV_z)\langle p^u\rangle\mu \\ d'N_2 &\approx dN_2 + \tau(V_xV_x + V_zV_z)\langle p^w\rangle\mu \end{aligned} \tag{8.17}$$

and hence the resulting solutions for $\tau \neq 0$ will have nonzero viscous forces, in general. The embedding thus gives one-parameter families of solutions of the Navier–Stokes equations that will be both new and interesting.

We can also start with a known exact solution of the Navier–Stokes equations. For example, take

$$\mathbf{U} = \frac{u_0}{a^2}(a^2 - z^2)\mathbf{i}, \qquad P = P_0 - \frac{2Mu_0}{a^2}x \tag{8.18}$$

which describes the steady flow of a viscous fluid between two fixed parallel plates that are situated at $z = \pm a$. The appropriate, completely integrable horizontal ideal $\mathscr{H}[A_{ij}^\alpha]$ is obtained by the assignment

$$A_{zz}^u = -\frac{2u_0}{a^2} \tag{8.19}$$

and all other A's set equal to zero. We therefore have the canonical basis vectors

$$\begin{aligned} V_x &= \partial_x + y_x^u\partial_u + y_x^w\partial_w + y_x^p\partial_p \\ V_z &= \partial_z + y_z^u\partial_u + y_z^w\partial_w + y_z^p\partial_p - \frac{2u_0}{a^2}\partial_u^z \\ V_t &= \partial_t + y_t^u\partial_u + y_t^w\partial_w + y_t^p\partial_p \end{aligned} \tag{8.20}$$

for $\mathscr{H}^*[A_{ij}^{\alpha}]$. A complete system of independent primitive integrals of the system $\{V_i\langle g\rangle = 0 | 1 \le i \le 3\}$ is given by

$$\begin{gathered} g^u = u - y_x^u x - y_z^u z - y_t^u t - \frac{u_0}{a^2} z^2 \\ g^w = w - y_x^w x - y_z^w z - y_t^w t, \qquad g^p = p - y_x^p x - y_z^p z - y_t^p t \\ g_x^u = y_x^u, \qquad g_z^u = y_z^u + \frac{2u_0}{a^2} z, \qquad g_t^u = y_t^u \\ g_x^w = y_x^w, \qquad g_z^w = y_z^w, \qquad g_t^w = y_t^w \\ g_x^p = y_x^p, \qquad g_z^p = y_z^p, \qquad g_t^p = y_t^p \end{gathered} \tag{8.21}$$

The solution leaf of the foliation generated by $\mathscr{H}[A_{ij}^{\alpha}]$ is therefore specified by

$$\begin{gathered} g^u = u_0, \qquad g_x^u = g_z^u = g_t^u = 0 \\ g^w = g_x^w = g_z^w = g_t^w = 0 \\ g^p = p_0, \qquad g_x^p = -\frac{2Mu_0}{a^2}, \qquad g_z^p = g_t^p = 0 \end{gathered} \tag{8.22}$$

The initial data for the evolution equations (8.14) that the $'A$'s must satisfy are therefore given by

$$'A_{ij}^u('z^A; 0) = -\frac{2u_0}{a^2} \delta_i^z \delta_j^z, \qquad 'A_{ij}^w('z^A; 0) = 'A_{ij}^p('z^A; 0) = 0 \tag{8.23}$$

A direct approach to solving the Navier-Stokes equations is provided by the method of extended Hamilton-Jacobi maps. Here, we consider extended canonical transformations of the form

$$\mathrm{HJ}|'x^i = x^i, \qquad 'q^{\alpha} = J^{\alpha}(z^A), \qquad 'y_i^{\alpha} = V_i\langle J^{\alpha}(z^A)\rangle \tag{8.24}$$

where $\{V_i | 1 \le i \le n\}$ is a canonical system associated with a completely integrable horizontal ideal $\mathscr{H}[A_{ij}^{\alpha}] \subset \Lambda(K_1)$. Such transformations are Hamilton-Jacobi transformations for the balance ideal of the Navier-Stokes equations if and only if HJ^* pulls the balance n-forms (8.4) back to zero mod $\mathscr{H}[A_{ij}^{\alpha}]$. This gives us the requirements

$$\begin{aligned} \hat{F}_1 = {} & V_t\langle J^u\rangle + J^u V_x\langle J^u\rangle + J^w V_z\langle J^u\rangle + V_x\langle J^p\rangle \\ & - M(V_x V_x + V_z V_z)\langle J^u\rangle = 0 \end{aligned} \tag{8.25}$$

$$\begin{aligned} \hat{F}_2 = {} & V_t\langle J^w\rangle + J^u V_x\langle J^w\rangle + J^w V_z\langle J^w\rangle + V_z\langle J^p\rangle \\ & - M(V_x V_x + V_t V_t)\langle J^w\rangle = 0 \end{aligned} \tag{8.26}$$

$$\hat{F}_3 = V_x\langle J^u\rangle + V_z\langle J^w\rangle = 0 \tag{8.27}$$

If we take the generating functions $\{s^u, s^w, s^p\}$ to be functions of the variables $\{x, z, t\}$ only, then (8.25)-(8.27) reduce to the original Navier-Stokes equations. Thus, any solution of the Navier-Stokes equations generates an extended Hamilton-Jacobi map for those equations that simply reproduces that known solution. For example, if we take $A^\alpha_{ij} = 0$, an inspection of (8.25)-(8.27) shows that a solution is given by

$$J^u = a(t) + bz + \tfrac{1}{2}cz^2 + \gamma(z, t), \qquad J^w = 0$$

$$J^p = p_0 + \left(Mc - \frac{da(t)}{dt} \right) x$$

for any smooth function $a(t)$ and any smooth function $\gamma(z, t)$ that satisfies the linear diffusion equation

$$\frac{\partial \gamma}{\partial t} = M \frac{\partial^2 \gamma}{\partial z^2}$$

If we take the generating functions $\{J^u, J^w, J^p\}$ to be functions of the variables $\{u, w, p, y^u_x, y^u_z, y^u_t, y^w_x, y^w_z, y^w_t, y^p_x, y^p_z, y^p_t\}$ only, then the system (8.25)-(8.27) becomes a new system of nonlinear, second-order PDE whose solutions will generate solutions of the Navier-Stokes equations by the corresponding Hamilton-Jacobi map of leaves of the foliation generated by the source horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$. For example, if we take

$$A^\alpha_{ij} = k^\alpha_{ij} = k^\alpha_{ji}, \qquad dk^\alpha_{ij} = 0 \tag{8.28}$$

then the operators $\{V_i | 1 \leq i \leq 3\}$ are determined and the q's on leaves of the foliation of K that are generated by $\mathscr{H}[A^\alpha_{ij}]$ are quadratic functions of the x's. Remembering that $[\![V_i, V_j]\!] = 0$, equation (8.27) has the general solution

$$J^u = V_x\langle\phi\rangle + V_z\langle\xi\rangle, \qquad J^w = V_z\langle\phi\rangle - V_x\langle\xi\rangle \tag{8.29}$$

for any smooth functions ϕ and ξ of the variables $\{u, w, \ldots, y^p_t\}$ such that

$$(V_x V_x + V_t V_t)\langle\phi\rangle = 0 \tag{8.30}$$

The relations (8.29) can then be put back into (8.25) and (8.26) to obtain a system of two PDE in the unknowns $\{\phi, \xi, s^p\}$. Since $[\![V_i, V_j]\!] = 0$, the unknown s^p can be eliminated from these two equations by cross application of V_x and V_z, and for $\phi = 0$, we obtain a single, nonlinear, fourth-order PDE

$$\begin{aligned} 0 = {} & [V_t - M(V_x V_x + V_z V_z)](V_x V_x + V_z V_z)\langle\xi\rangle \\ & + (V_z\langle\xi\rangle V_x - V_x\langle\xi\rangle V_z)(V_x V_x + V_z V_z)\langle\xi\rangle \end{aligned} \tag{8.31}$$

for the determination of ξ. We note that this equation becomes linear if either $V_x\langle\xi\rangle = 0$ or $V_z\langle\xi\rangle = 0$, and hence it is easily solved in these two cases.

Any solution of (8.31) can be put back into (8.25) and (8.26) and s^p can then be determined by quadratures. The extended Hamilton–Jacobi method can thus provide a large class of interesting solutions to the Navier–Stokes equations.

A simpler choice for the A's is

$$A^u_{ij} = A^w_{ij} = A^p_{ij} = 0 \tag{8.32}$$

in which case the q's become linear functions of the x's on the leaves of the foliation of K that is generated by $\mathcal{H}[A^\alpha_{ij} = 0]$. If we set

$$J^w = 0 \tag{8.33}$$

the system (8.25)–(8.27) reduces to

$$V_x\langle J^u\rangle = 0, \qquad V_z\langle J^p\rangle = 0 \tag{8.34}$$

$$V_t\langle J^u\rangle + V_x\langle J^p\rangle = MV_zV_z\langle J^u\rangle \tag{8.35}$$

If we introduce the new variables

$$\xi \equiv y^u_x w - y^w_x u, \qquad \eta \equiv y^u_x p - y^p_x u \tag{8.36}$$

$$\alpha = y^u_z w - y^w_z u, \qquad \beta = y^u_z p - y^p_z u \tag{8.37}$$

then the equations (8.34) are satisfied by

$$J^u = \mathcal{U}(\xi, \eta; y^\gamma_1), \qquad J^p = \mathcal{P}(\alpha, \beta; y^\gamma_i) \tag{8.38}$$

Noting that any three of the variables $\{\xi, \eta, \alpha, \beta\}$ are independent, for y's in general position, (8.35) can be satisfied only if

$$J^p = a(y^\gamma_j)\alpha + b(y^\gamma_j)\beta \tag{8.39}$$

in which case (8.35) reduces to

$$\begin{aligned}(y^w_t y^u_x - y^w_x y^u_t)\frac{\partial u}{\partial \xi} &+ (y^p_t y^u_x - y^p_x y^u_t)\frac{\partial u}{\partial \eta} \\ &- M\left\{(y^w_z y^u_x - y^w_x y^u_z)\frac{\partial}{\partial \xi} + (y^p_z y^u_x - y^p_x y^u_z)\frac{\partial}{\partial \eta}\right)^2\langle \mathcal{U}\rangle \\ &= a(y^\gamma_j)(y^w_z y^u_x - y^w_x y^u_z) + b(y^\gamma_j)(y^p_z y^u_x - y^p_x y^u_z)\end{aligned} \tag{8.40}$$

Since the y's are parametric variables, (8.40) is a linear, inhomogeneous, second-order PDE that is readily solvable for the function $\mathcal{U}(\xi, \eta; y^\gamma_j)$. We therefore have the Hamilton–Jacobi maps of the Navier–Stokes equations that are generated by

$$J^u = \mathcal{U}(y^u_x w - y^w_x u, y^u_x p - y^p_x u; y^\gamma_j), \qquad J^w = 0 \tag{8.41}$$

$$J^p = a(y^\gamma_j)(y^u_z w - y^w_z u) + b(y^\gamma_j)(y^u_z p - y^p_z u) \tag{8.42}$$

for every choice of the functions $a(y_j^\gamma)$, $b(y_j^\gamma)$ and for every function $\mathcal{U}(\xi, \eta; y_j^\gamma)$ that satisfies (8.40). Since $'w = J^w = 0$, the Hamilton–Jacobi map defined by (8.41) and (8.42) is not an element of Diff$(K, 'K)$.

REFERENCES

Bäcklund, A. V. (1976). Uber Flächentransformationen, *Mathematische Annalen*, **9**, 297–320.

Edelen, D. G. B. (1990). *International Journal of Theoretical Physics*, **29**, 687–737.

Ibragimov, N. H. (1985). *Transformation Groups Applied to Mathematical Physics*, Reidel, Boston.

Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*, Springer-Verlag, Berlin.

Ovsiannikov, L. V. (1982). *Group Analysis of Differential Equations*, Academic Press, New York.

Pommaret, J. F. (1978). *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach, New York.

Rogers, C., and Shadwick, W. F. (1982). *Bäcklund Transformations and Their Applications*, Academic Press, New York.